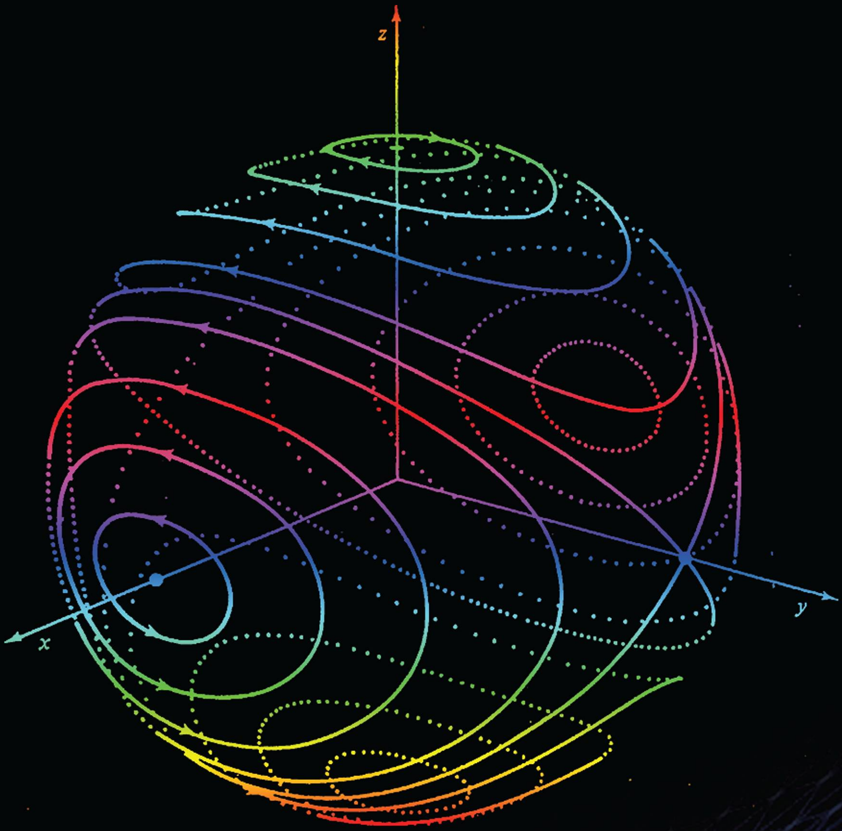
G. S. CHADDHA

Classical Mechanics



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CLASSICAL MECHANICS

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PREFACE

This book has evolved from my teaching of a course in classical mechanics to M. Sc. students at Pant University of A&T, Pantnagar. I have taught this one-semester (4 credits) course for several years, each time becoming more convinced of the need for a new text book in this pretty important area.

My overall selection of topics and solutions to almost all problems at the end of each chapter completely fulfil the requirements of Indian Universities/Institutions in particular and of foreign universities/institutions in general, though the duplication of the standard material is impossible to avoid.

The book is intended primarily for students of physics at M.Sc./BS/MS levels, but it might also be used successfully by students in other disciplines, such as engineering and mathematics. It of course assumed that the reader has a thorough familiarity with the basic principles of mechanics and with integral and differential calculus.

The book comprises of nine chapters and four appendices. The first chapter deals with Lagrange's equations of motion and the 2nd with some aspects of the calculus of variations. The 3rd chapter deals with the integration of the equation of motion in one-dimension, motion in central force field and the scattering problems. The fourth chapter deals with the motion of a rigid body, angular momentum, inertia tensor, Euler's equations of motion and with the motion of a heavy symmetrical top. The fifth chapter deals with the theory of small oscillations in the cases of one and more than one degrees of freedom. A detailed discussion on the normal coordinates is also given. Chapters 6 and 7 deal with Hamilton's equations of motion, cyclic co-ordinates, conservation theorems, principle of least action, canonical transformations and Poisson brackets. Chapter 8 deals with Hamilton-Jacobi theory and action-angle variables. The last chapter 9 deals with the Lagrangian and Hamiltonian formulation for continuous systems with applications to longitudinal vibrations of the particles of the elastic rod and transverse vibrations in the case of a stretched string. As pointed out earlier end of almost each chapter contains a problem set. Most of the problems (about 74 in total apart from the solved examples) are solved so that a reader can have an indepth knowledge of the subject. Appendices A, B, C and D truly speaking complete the discussion in the text.

The author is highly thankful to his teachers, Dr. P.K. Sharma, while he was a M.Sc. Student at Allahabad University and to Prof. Mathews (a visiting Professor from U.S.A.) while he was a Ph.D. Scholar at Indian Institute of Technology, Kanpur. Thanks are also due to all my students who made me to go into the deep understanding of the subject.

(Gurbachan S. Chaddha)

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CHAPTER 1

Lagrange's Equations

1.1 CONSTRAINTS

For a single particle, the equation of motion (or Newton's Second law) is given by

$$\vec{F} = \dot{\vec{p}} \tag{1}$$

where $\vec{F}$ is the total force (vector sum of forces, *viz*. gravitational or electrodynamic) exerted on the particle and $\vec{p}$ is the linear momentum of the particle.

For a system consisting of particles, the equation of motion for the ith particle is, therefore written as

$$\sum_j{}' \vec{F}_{ji} + \vec{F}_i^{(e)} = \dot{\vec{p}} = m_i \ddot{\vec{r}}_i \tag{2}$$

where $\vec{F}_i^{(e)}$ stands for an external force acting on the ith particle because of the sources outside the system and $\vec{F}_{ji}$ is the internal force on the ith particle due to the jth particle. $\sum_j{}'$ indicates the exclusion of the term $j = i$, that is, $\vec{F}_{ii}$ as it is evidently zero.

One may infer from the above that all problems in mechanics have been reduced to, solving the set of differential equs. (2). But this is not so. Truly speaking one has to take into account the constraints (restrictions) that limit the motion of the system. One simple example of the constrained system is that of the rigid body where the constraints on the motions of the particles are such that the interparticle distances r_{ij} remain unchanged with respect to time. There are several other examples of the constrained systems. The beads of an abacus are constrained to one-dimensional motion by the supporting wires. Gas molecules within a container are constrained to move only inside the container by the wall of the vessel. A particle placed on the surface of a solid sphere is constrained to move only on the surface or in the region exterior to the sphere.

Constraints may be classified in various ways. If constraints or the conditions of constraint can be expressed in the form of equations

$$f_k(\vec{r}_1, \vec{r}_2, \ldots\ldots\ldots\ldots\ldots\ldots, \vec{r}_N, t) = 0 \tag{3}$$

relating the coordinates of the particles (including possibly the time if the constraints are changing with time), they are then said to be holonomic. Here we are considering a system of N particles subject to k independent constraints expressed in k equations. $\vec{r}_1, \vec{r}_2, \ldots\ \vec{r}_N$ are the position vectors of the N particles.

The simplest example of the holonomic constraints is that of the rigid body where the constraints are expressed by equations of the form

$$|\vec{r}_i - \vec{r}_j|^2 - c_{ij}^2 = 0 \tag{4a}$$

where $|\vec{r}_i - \vec{r}_j| = |\vec{r}_{ij}|$ denotes distance between ith and jth particles and c_{ij} are constants. In terms of their cartesian coordinates, the conditions of constraint can be expressed as

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - c_{ij}^2 = 0 \tag{4b}$$

A particle constrained to move along any curve or surface is another example of holonomic constraint. If the particle is confined to a circular path of radius a in the xy plane, its coordinates would have to obey the two equations,

$$x^2 + y^2 - a^2 = 0 \text{ and } z = 0 \tag{4c}$$

Constraints which can not be expressed in the above fashion (or in the form of equs. 3) are called nonholonomic. The constraint involved in the example of a particle placed on the surface of a sphere is nonholonomic as the condition of constraint can be expressed by an inequality

$$r^2 - a^2 \geq 0 \tag{5}$$

which is not in the form of equ. (3). a is the radius of the sphere and r the position of the particle. In a gravitational field a particle placed on the top of sphere will roll (or slide) down the surface part of the way but will finally fall off. The equality sign holds until the particle rolls on the surface.

Constraints are further classified according as the equations of constraint contain the time explicitly (rheonomous) or are not explicitly dependent on time (scleronomous) or in the other words are independent of time. Therefore, the constraint in the case of a bead sliding on a wire fixed in space is scleronomous while that of a bead on the wire moving in some prescribed fashion, is rheonomous.

It may be noted here that the constraints may be expressed in the form of equations connecting the velocities rather than in the form of equations connecting the coordinates. But if a constraint on the velocities can be integrated to give a relation between the coordinates, then the constraint is still said to be holonomic. For example, a cylinder of radius a rolling down an inclined plane, with its axis always horizontal, can be located by two coordinates s measuring the distance the cylinder has moved down the plane and θ measuring the angle that a fixed radius in the cylinder has rotated from the radius to the point of contact with the plane (see Fig. 1.1). Supposing that the cylinder is rolling without slipping, the velocities $\dot{s}$ (linear) and $\dot{\theta}$ (angular) would be related by the constraint equation

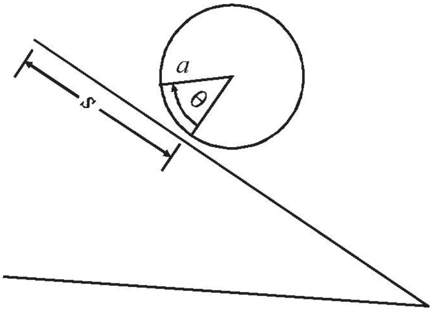

Fig. 1.1 A cylinder rolling down an inclined plane

$$\dot{s} = a\dot{\theta} \tag{6a}$$

which can also be written as

$$ds = ad\theta \tag{6b}$$

The above equation after integration reduces to

$$s - a\theta = \text{constt.} \tag{7}$$

which is a relation between coordinates like equs. (3) or (4).

There are systems, however, in which equations of constraint (relation between velocities) can not be integrated or in other words the constraints can not be reduced to the form (3) and are therefore nonholonomic. A simple case would illustrate this point. Consider a disk of radius a rolling on the horizontal table (xy plane) such that the diameter which touches the table is always vertical (see Fig. 1.2). Four coordinates are required to specify position of the disk. The coordinates x, y locate the point of contact of the disk on the plane, the angle ϕ describes the orientation of the plane of the disk relative to the x-axis and the angle θ is the angle of rotation about the axis of the disk. If we now require that the disk rolls without slipping, then as a result of the constraint, the velocity of the point of contact

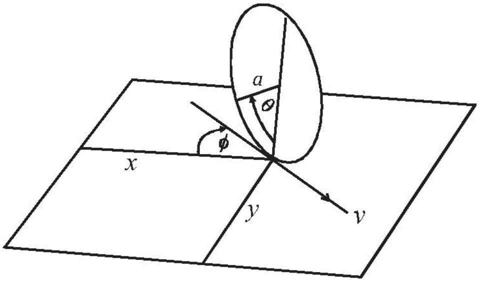

Fig. 1.2 A disk rolling on a horizontal plane

$$v = a\dot{\theta} \tag{8}$$

and its direction is perpendicular to the axis of the disk. Therefore

$$\dot{x} = v \cos \phi$$

and $$\dot{y} = -v \sin \phi$$

Substituting the value of v, one gets

$$\dot{x} = a\, \dot{\theta} \cos \phi$$

and $$\dot{y} = -a\, \dot{\theta} \sin \phi$$

From above, one gets the following two equations of constraint:

$$dx - a \cos\phi\, d\theta = 0 \qquad (9)$$

$$dy + a \sin\phi\, d\theta = 0$$

Neither of equs. (9) can be integrated or in other words it is not possible to get relations between the coordinator x, y, θ and ϕ. This implies that the constraints are nonholonomic. Qualitatively to see why the above happens, one may note that by rolling the disk without slipping, and by rotating it about a vertical axis, one can bring the disk to any point x, y with any angle ϕ between the plane of the disk and x-axis and with any point on the circumference of the disk in contact with the table, that is, with any angle θ. This shows that the four coordinates x, y, θ and ϕ are independent of one another and thus there can not be any relation between them.

1.2 GENERALIZED COORDINATES

Let us consider a system of N particles. To define the positions of particles in the system (or configuration of the system), free from constraints, it is necessary to have 3 N independent cartesian coordinates $x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_N, y_N, z_N$ or N radii vectors $\vec{r}_1, \vec{r}_2, \ldots. \vec{r}_N$ (each one having three components). Since the number of independent quantities (or coordinates) which must be specified in order to define uniquely the positions of particles in any system is called the number of degrees of freedom, it is 3N here. If there exist holonomic constraints, expressed in k equations in the form (3), then one may use these equations to eliminate k of the 3N coordinates, and one would be left with 3N-k independent coordinates and the system is therefore, said to have 3N-k degrees of freedom. These independent quantities need not be cartesian coordinates of the particles. Truly speaking the conditions of the problem may render some other choice of coordinates more suitable. Any s (=3N-k) independent quantities or variables $q_1, q_2, \ldots, q_s$ which completely define the positions of particles in the system with s degrees of freedom are called generalized coordinates of the system and the time derivatives $\dot{q}_i$ are called its generalized velocities.

Since for each configuration of the system, the generalized coordinates must have some definite set of values, the coordinates $q_1, q_2, \ldots, q_s$ would be functions of the old coordinates $\vec{r}_1, \vec{r}_2, \ldots. \vec{r}_N$ and possibly also of the time in the case of moving coordinate systems. That is,

$$q_1 = q_1(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N, t) \tag{10}$$
$$\vdots$$
$$q_s = q_s(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N, t)$$

These are transformation equations from the set of (q_j) variables to the $(\vec{r}_i)$ set or alternatively equs. (10) can be considered as parametric representation of the (q_j) variables. Since the coordinates $q_1, q_2, \ldots, q_s$ specify the configuration of the system, it must also be possible to express the old coordinates $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N$ in terms of the generalized coordinates. That is,

$$\vec{r}_1 = \vec{r}_1(q_1, q_2, \ldots, q_s, t) \tag{11}$$
$$\vdots$$
$$\vec{r}_N = \vec{r}_N(q_1, q_2, \ldots, q_s, t)$$

As an example, we consider the motion of a single particle in a plane with moving coordinate system. For such a system, we consider polar coordinates in which the reference axis from which θ is measured rotates anticlockwise with constant angular velocity ω (see Fig. 1.3). The polar coordinates r, θ are related to the cartesian coordinates x, y by the following equations.

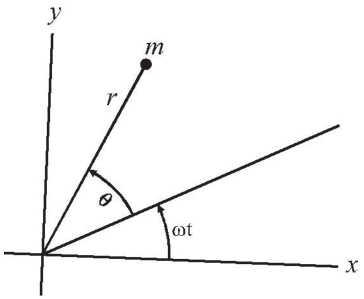

Fig. 1.3 A rotating polar coordinate system

$$r = (x^2 + y^2)^{1/2},$$
$$\tan(\omega t + \theta) = y/x \tag{12}$$

or
$$\theta = \tan^{-1} y/x - \omega t$$

and conversely

$$x = r\cos(\omega t + \theta) \text{ or } x = x(r, \theta, t),$$
$$y = r\sin(\omega t + \theta) \text{ or } y = y(r, \theta, t) \tag{13}$$

Note that in the above quoted example, there are as many generalized coordinates (q_1 and q_2 in the present case) as the cartesian coordinates (x, y) or polar coordinates (r, θ). That is,

$$q_1 = x, \qquad q_1 = r = (x^2 + y^2)^{1/2},$$
$$\text{or}$$
$$q_2 = y, \qquad q_2 = \theta = \tan^{-1} y/x - \omega t \tag{14}$$

Consider the motion of a simple pendulum (see Fig. 1.4) oscillating in a vertical (xy) plane. Its motion can be described in terms of the two cartesian coordinates x and y given by

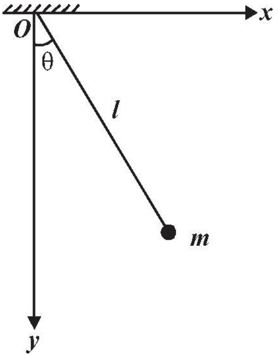

Fig. 1.4 A simple pendulum oscillating in the x-y plane

$$x = l \sin\theta, y = l \cos\theta$$

Since l is constant, the only variable involved to locate the oscillating mass is θ, which can be chosen as the generalized coordinate. Therefore,

$$q = \theta = \sin^{-1} x/l \text{ or } \cos^{-1} y/l \tag{15}$$

1.3 D' ALEMBERT'S PRINCIPLE AND LAGRANGE'S EQUATIONS

A virtual displacement of a system refers to a change in the configuration of the system as a result of any arbitrary infinitesimal change of coordinates $\delta\vec{r}_i$ consistent with the forces and constraints imposed on the system at a given instant t. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in time interval dt during which the forces and constraints may be changing. Suppose the system is in equilibrium, that is, the total force acting on each particle vanishes, $\vec{F}_i = 0$. Then the virtual work done by the force $\vec{F}_i$ in the displacement $\delta\vec{r}_i$, that is, $\vec{F}_i . \delta\vec{r}_i$, also vanishes. The sum of these vanishing products over all particles must likewise be zero:

$$\sum_{i=1}^{N} \vec{F}_i . \delta\vec{r}_i = 0 \tag{16}$$

Taking into account the equation of motion of the ith particle

$$\vec{F}_i = \dot{\vec{p}}_i \text{ (or } \vec{F}_i - \dot{\vec{p}}_i = 0),$$

instead of equ. (16), we can write

$$\sum_{i=1}^{N} (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta\vec{r}_i = 0 \tag{17}$$

Writing the total force $\vec{F}_i$ as the sum of the applied force $\vec{F}_i^a$ and the force of constraint, $\vec{f}_i$, that is

$$\vec{F}_i = \vec{F}_i^a + \vec{f}_i,$$

equ. (17) becomes

$$\sum_{i=1}^{N}(\vec{F}_i - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i + \sum_{i=1}^{N}\vec{f}_i\cdot\delta\vec{r}_i = 0 \tag{18}$$

Let us now restrict ourselves to systems for which the total virtual work of the forces of constraint vanishes. An example of such a system is a particle constrained to move on a smooth surface. The force of constraint (Reaction) being perpendicular to the surface in this case while virtual displacement being tangential to it, the virtual work of the force of constraint vanishes. This is no longer true in case the frictional force is present, which would be tangential to the surface and hence the virtual work of the said force would not be zero. We, therefore, exclude such systems from our formulation. This restriction does not create any problem since the friction is essentially a macroscopic phenomenon. Therefore, in the present situation equ. (18) becomes

$$\sum_{i=1}^{N}(\vec{F}_i^a - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i = 0 \tag{19}$$

which is often called D'Alembert's principle. Since the forces of constraint no longer appear, we can drop the superscript a, and can regard $\vec{F}_i$ as the applied (external) force. As a result the above equation can be written as

$$\sum_{i=1}^{N}(\vec{F}_i - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i = 0 \tag{19a}$$

The principle written in the above form is still not useful for furnishing equations of motion for the system involving holonomic constraints as $\delta\vec{r}_i$ are not independent of each other. We must, therefore, transform the principle into an expression involving virtual displacements of the generalized coordinates, which are independent of each other, so that the coefficients of the δq_j can be set separately equal to zero.

The transformation from $\vec{r}_i$ set to q_j set is carried out by equs. (11),

$$\vec{r}_i = \vec{r}_i\,(q_1, q_2, \ldots\ldots, q_s, t),\ i = 1, 2, \ldots, N$$

Using above relation, the first term of equ. (19a), that is, the virtual work of $\vec{F}_i$ in terms of the generalized coordinates becomes

$$\sum_{i=1}^{N}\vec{F}_i\cdot\delta\vec{r}_i = \sum_{i=1}^{N}\vec{F}_i\cdot\left[\sum_j\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j + \frac{\partial\vec{r}_i}{\partial t}\delta t\right],\ j = 1, 2, \ldots..s$$

One must recollect that there is no variation of time, that is, $\delta t = 0$ for a virtual displacement. As a result, the last term in the above expression vanishes and we are left with

$$\sum_{i=1}^{N}\vec{F}_i\cdot\delta\vec{r}_i = \sum_{i,j}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = \sum_{j}Q_j\,\delta q_j$$

where $$Q_j = \sum_{j}\vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j\,, j = 1, 2, ...,s \qquad (20a)$$

are called the components of the generalized force (it would have dimensions of force if and only if, q_j is a length coordinate. If q_j is an angle, in that case Q_j would be a torque. But $Q_j\,\delta q_j$ must always have the dimensions of work)

Let us now turn to the second term in equ. (19a) which may be written as

$$\begin{aligned}\sum_{i=1}^{N}\dot{\vec{p}}_i\cdot\delta\vec{r}_i &= \sum_{j}m_i\ddot{\vec{r}}_i\cdot\delta\vec{r}_i\\ &= \sum_{i,j}m_i\,\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j\\ &= \sum_{i,j}\left[\frac{d}{dt}(m_i\,\dot{\vec{r}}_i\,.\frac{\partial\vec{r}_i}{\partial q_j}) - m_i\,\dot{\vec{r}}_i\,.\frac{d}{dt}(\frac{\partial\vec{r}_i}{\partial q_j})\right]\delta q_j\\ &= \sum_{i,j}\left[\frac{d}{dt}(m_i\vec{v}_i.\frac{\partial\vec{r}_i}{\partial q_j}) - m_i\vec{v}_i\,.\frac{d}{dt}(\frac{\partial\vec{r}_i}{\partial q_j})\right]\delta q_j \qquad (21)\end{aligned}$$

In the last term of the above equation, we can interchange the differentiations with respect to t and q_j†. Doing so, we get

$$\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d\vec{r}_i}{dt}\right) = \frac{\partial}{\partial q_j}\cdot\vec{v}_i \qquad (20b)$$

Again using the transformation equs. (11), we get

$$\frac{d\vec{r}_i}{dt} = \vec{v}_i = \sum_{j}\frac{\partial\vec{r}_i}{\partial q_j}q_j + \frac{\partial\vec{r}_i}{\partial t} \qquad (20c)$$

Differentiating the above equation *w.r.t.* $\dot{q}_j$, we get

---

† $$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\vec{r}_i}{\partial q_j}\right) &= \sum_{l=1}^{s}\frac{\partial}{\partial q_j}\left(\frac{\partial\vec{r}_i}{\partial q_j}\right)\dot{q}_l + \frac{\partial}{\partial t}\left(\frac{\partial\vec{r}_i}{\partial q_j}\right)\\ &= \sum_{l}\frac{\partial^2\vec{r}_i}{\partial q_l\partial q_j} + \frac{\partial^2\vec{r}_i}{\partial t\partial q_j}\end{aligned}$$

also $$\begin{aligned}\frac{\partial}{\partial q_j}\left(\frac{d\vec{r}_i}{dt}\right) &= \frac{\partial}{\partial q_j}\left[\sum_{l}\frac{\partial^2\vec{r}_i}{\partial q_l}\dot{q}_l + \frac{\partial\vec{r}_i}{\partial t}\right]\\ &= \sum_{l}\frac{\partial^2\vec{r}_i}{\partial q_j\partial q_l} + \frac{\partial^2\vec{r}_i}{\partial q_j\partial t}\end{aligned}$$

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \tag{20d}$$

as the second term in equ. (20c) is not a function of q_j. Also summation over j in the first term of the said equation would contribute only one term because when the differentiation with respect to $\dot{q}_j$ is found, all other $\dot{q}$'s would be regarded as constant.

The substitution of equs. (20b) and (20d) into equ. (21) gives

$$\begin{aligned} \sum_{i=1}^{N} \dot{\vec{p}}_i \cdot \delta \vec{r}_i &= \sum_{j=1}^{s} \left[\frac{d}{dt} \left(\sum_{i=1}^{N} m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - \left(\sum_{i=1}^{N} m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right) \right] \delta q_j \\ &= \sum_{j=1}^{s} \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} \left(\sum_{i=1}^{N} \tfrac{1}{2} m_i v_i^2 \right) - \frac{\partial}{\partial q_j} \left(\sum_{i=1}^{N} \tfrac{1}{2} m_i v_i^2 \right) \right\} \right] \delta q_j \\ &= \sum_{j=1}^{s} \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right] \delta q_j \end{aligned} \tag{22}$$

where T is the kinetic energy of the system of particles.

Combining equs. (20a) and (22) leads to D'Alembert's principle (equ. 19a) in the form

$$\sum_{j=1}^{s} \left[\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0 \tag{23}$$

Since the constraints are holonomic, the q's and δq's are independent of one another. Therefore, equ. (23) holds true only if all the square brackets are zero, that is

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j, j = 1, 2, ..., s \tag{24}$$

The above set of s equations are often referred to as Lagrange's equations but this name is used mostly for the form of equs. (24) when the forces involved are conservative or in other words are derivable from a scalar potential function V called the potential energy of the system (which does not depend explicitly on time or on the generalized velocities $\dot{q}$) which depends only on positions. These assumptions are usually valid. Then

$$\vec{F}_i = -\vec{\nabla}_i V (\text{or} - \partial V / \partial \vec{r}_i) \tag{25}$$

In the above case the generalized forces can be written as

$$Q_j = \sum_j \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \frac{\partial V}{\partial \vec{r}_i} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\partial V(\vec{r}_1, \vec{r}_2, ...\vec{r}_N) / \partial q_j \tag{26}$$

Since $\partial V / \partial \dot{q}_j = 0$ as V is independent of the generalized velocities, the equs. (24) can be written as

$$\frac{d}{dt}\frac{\partial(T-V)}{\partial \dot{q}_j}-\frac{\partial(T-V)}{\partial q_j}=0 \tag{27}$$

Defining a new function, the Lagrangian L as

$$\begin{aligned} L &= L(q_1, q_2, \ldots\ldots\ldots, q_s;\ \dot{q}_1, \dot{q}_2, \ldots\ldots\ldots\ldots\ldots\ldots, \dot{q}_s, t) \\ &= T-V, \end{aligned} \tag{27'}$$

the equs. (24) become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=0, \quad j=1,2,\ldots,s \tag{28}$$

which are known as Largrange's equations of motion for conservative systems. As j ranges from 1 to s, there would be s such 2nd order differential equations in the variables q_j.

1.4 VELOCITY - DEPENDENT POTENTIAL

Let us show that the Lagrange's equations can be put in the form (28) in the case of a potential $U(q_j, \dot{q}_j)$, linear in $\dot{q}_j$ if the generalized forces can be taken as

$$Q_j=-\frac{\partial U}{\partial q_j}+\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j}, \quad j=1,2,\ldots,s \tag{29}$$

where U is called generalized or velocity - dependent potential. An example of such a potential is offered by the motion of an electrically charged particle in an electromagnetic field.

Putting the above value of Q_j in equ. (24), one gets

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}=-\frac{\partial U}{\partial q_j}+\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j}$$

or
$$\frac{d}{dt}\frac{\partial(T-U)}{\partial \dot{q}_j}-\frac{\partial(T-U)}{\partial q_j}=0$$

If we take the Lagrangian in such a case to be given by

$$L=T-U, \tag{30a}$$

we get
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=0, \quad j=1,2,\ldots\ldots,s \tag{30b}$$

Let us now consider the case of an electrically charged particle moving in an electromagnetic field. In such a case, the electromagnetic force, called the Lorentz force acting on a particle of mass m and charge e, moving with velocity $\vec{v}$, is given by

$$\vec{F}=e\{\vec{E}+\vec{v}\times\vec{B}\} \tag{31}$$

Let us write the following two Maxwell's equations

$$\vec{\nabla}\times\vec{E}+\partial\vec{B}/\partial t = 0 \tag{32}$$

and

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{33}$$

Since $\vec{\nabla}\times\vec{E}\neq 0$, hence $\vec{E}$ can not be gradient of a scalar function, but from $\vec{\nabla}\cdot\vec{B}$ = 0, it follows that $\vec{B}$ could be represented as curl of a vector, that is

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{34}$$

where $\vec{A}$ is called the magnetic vector potential. Substituting the value of $\vec{B}$ in equ. (32), one gets

$$\vec{\nabla}\times\vec{E}+\partial/\partial t\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0$$

Hence, we can write

$$\vec{E}+\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$$

or

$$\vec{E} = -\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t} \tag{35}$$

where ϕ is given the name scalar potential.

In terms of the potentials $\vec{A}$ and ϕ, the Lorentz force becomes

$$\vec{F} = e\left[-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}+\vec{\upsilon}\times\vec{\nabla}\times\vec{A}\right] \tag{36}$$

Now using the vector identify[††]

$$\vec{\upsilon}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}\left(\vec{\upsilon}\cdot\vec{A}\right)-\left(\vec{\upsilon}\cdot\vec{\nabla}\right)\vec{A},$$

one gets

$$\vec{F} = e\left[-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}+\vec{\nabla}\left(\vec{\upsilon}\cdot\vec{A}\right)-\left(\vec{\upsilon}\cdot\vec{\nabla}\right)\vec{A}\right] \tag{37}$$

As

$$\vec{A}\,(\vec{r}\,,t) = \vec{A}\,(x, y, z, t)$$

therefore,

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t}+\frac{\partial\vec{A}}{\partial x}\dot{x}+\frac{\partial\vec{A}}{\partial y}\dot{y}+\frac{\partial\vec{A}}{\partial z}\dot{z}$$

$$= \frac{\partial\vec{A}}{\partial t}+\left(\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\dot{z}\frac{\partial}{\partial z}\right)\vec{A}$$

$$= \frac{\partial\vec{A}}{\partial t}+(\dot{\vec{r}}\cdot\vec{\nabla})\vec{A}=\frac{\partial\vec{A}}{\partial t}+(\vec{\upsilon}\cdot\vec{\nabla})\vec{A}$$

or

$$\frac{\partial\vec{A}}{\partial t} = \frac{d\vec{A}}{dt}-\left(\vec{v}\cdot\vec{\nabla}\right)\vec{A} \tag{38}$$

[††] $\vec{a}\times\vec{b}\times\vec{c}=\vec{b}(\vec{a}\cdot\vec{c})-(\vec{a}\cdot\vec{b})\vec{c}$

Substituting the value of $\partial\vec{A}/\partial t$ from equ. (38) into equ. (37), one gets

$$\vec{F} = e\left[-\vec{\nabla}\phi - \frac{d\vec{A}}{dt} + \vec{\nabla}(\vec{\upsilon}\cdot\vec{A})\right]$$

$$= e\left[-\vec{\nabla}(\phi - \vec{\upsilon}\cdot\vec{A}) - \frac{d\vec{A}}{dt}\right]$$

Since the scalar potential ϕ is independent of velocity, the above expression is equivalent to

$$\vec{F} = e\left[-\vec{\nabla}(\phi - \vec{\upsilon}\cdot\vec{A}) + \frac{d}{dt}\frac{\partial}{\partial\vec{\upsilon}}(\phi - \vec{\upsilon}\cdot\vec{A})\right] \tag{39}$$

Thus the x-component of the Lorentz force is given by

$$F_x = e\left[-\frac{\partial}{\partial x}(\phi - \vec{\upsilon}\cdot\vec{A}) + \frac{d}{dt}\frac{\partial}{\partial\upsilon_x}(\phi - \vec{\upsilon}\cdot\vec{A})\right] \tag{40}$$

$$= -\frac{\partial}{\partial x}U + \frac{d}{dt}\frac{\partial}{\partial\upsilon_x}U \tag{40a}$$

where $$U = e\phi - e\vec{\upsilon}\cdot\vec{A} \tag{40b}$$

is called the generalized or velocity dependent potential.

Since the force on a charge e is not entirely given by the electric force $-e\vec{\nabla}\phi$ instead is derivable from a potential dependent on velocity, such a system is called non-conservative. Therefore, the Lagrangian for a charged particle moving in an electromagnetic field can be written as

$$L = T - U = T - e\phi + e\vec{\upsilon}\cdot\vec{A} \tag{41}$$

Example 1. A particle moves in a plane under the influence of a force acting toward a centre of force, whose magnitude is

$$F = 1/r^2\left(1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right)$$

and which truly speaking represents the force between two charges in Weber's electrodynamics. r is the distance of the particle to the centre of force. Find the generalized potential that would result in such a force and from that the Lagrangian for the motion of the particle in a plane.

In the problem under consideration, the generalized force is given by

$$F = -\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial\dot{r}} = 1/r^2\left(1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right)$$

Therefore, the generalized potential

$$U = 1/r\,(1 + \dot{r}^2/c^2)$$

$$\therefore \quad L = T - U = \tfrac{1}{2}m\,(\dot{r}^2 + r^2\dot{\theta}^2) - 1/r\,(1 + \dot{r}^2/c^2)$$

1.5 RAYLEIGH'S DISSIPATION FUNCTION

It may be noted here that if some of the forces (not all) acting on the system are derivable from a potential, then Lagrange's equations can always be written in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j, \quad j = 1, 2, \ldots\ldots\ldots\ldots, s \tag{42}$$

where L contains the potential of the conservative forces while Q_j represents the forces that are not derivable from a potential or called non-potential forces. Such a situation often occurs when the frictional or dissipative forces are present. Frictional forces are found to be proportional to the velocity of the particle and can be written as

$$\vec{F}_i^f = -k\vec{\upsilon}_i \qquad (i = 1, 2, \ldots\ldots\ldots, N; k > 0) \tag{43}$$

These forces are derivable from Rayleigh's dissipation function defined as

$$R = ½k\sum_{i=1}^{N}\upsilon_i^2 \tag{44}$$

From the above definition it is obvious that

$$\vec{F}_i^f = -\frac{\partial R}{\partial \vec{\upsilon}_i} = -\vec{\nabla}_{\upsilon_i} R \qquad (i = 1, 2, \ldots\ldots\ldots, N) \tag{45}$$

where $\vec{\nabla}_{\upsilon_i}$ stands for the partial derivative *w.r.t.* $\vec{\upsilon}_i$

To explain the physical significance of the Rayleigh's dissipation function, let us calculate the power dissipated by frictional forces. This is given by

$$\begin{aligned} P &= \sum_i \vec{F}_i^f \cdot \vec{\upsilon}_i \\ &= -k\sum_i \upsilon_i^2 \\ &= -2R, \end{aligned}$$

that is, the function R is equal to half of the power dissipated by friction.

The generalized forces Q_j^f arising from the frictional forces $\vec{F}_i^f$ are given by

$$\begin{aligned} Q_j^f &= \sum_i \vec{F}_i^f \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \vec{\nabla}_{\upsilon_i} R \cdot \frac{\partial \vec{r}_i}{\partial q_j} \\ &= -\sum_i \vec{\nabla}_{\upsilon_i} R \cdot \frac{\partial \vec{\upsilon}_i}{\partial \dot{q}_j} \end{aligned}$$

where we have used equ. (20 d)

or
$$Q_j^f = -\sum_i \frac{\partial \vec{\upsilon}_i}{\partial \dot{q}_j} \cdot \vec{\nabla}_{\upsilon_i} R = -\frac{\partial}{\partial \dot{q}_j} R, \; j = 1, 2, \ldots\ldots\ldots, s \tag{46}$$

Substituting the above value of $Q_j = Q_j^f$ in equ. (42), the Lagrange's equations become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial R}{\partial \dot{q}_j} = 0, \qquad j = 1, 2, \ldots\ldots, s \tag{47}$$

Therefore, two scalar functions L and R must be specified to obtain the equations of motion, if frictional forces are acting on the system.

1.6 HAMILTON'S PRINCIPLE AND LAGRANGE'S EQUATIONS

The derivation of Lagrange's equations presented in section. 1.3 was started from a consideration of the instantaneous state of the system and small virtual displacements about the instantaneous state, that is, from a differential principle such as D'Alembert's principle. It is also possible to obtain Lagrange's equations from a principle which considers the entire motion of the system between times t_1 and t_2 and small virtual variations of the entire motion from the actual motion. A principle of this nature is known as integral principle (or Hamilton's principle or some times named the principle of least action). The principle was published in 1834 by William Rowan Hamilton.

Before presenting the integral principle, let us first understand the meaning attached to the motion of the system between times t_1 and t_2. The instantaneous configuration of a system is described by the values of the s generalized coordinates $q_1, q_2, \ldots, q_s$ and corresponds to a particular point (the way x, y, z corresponds to a point in the three dimensional cartesian space) in a cartesian hyperspace where the q's form the s coordinate axes. This s-dimensional space, therefore, is known as configuration space. As time goes on, the state of the system changes and the system point moves in the configuration space, tracing out a curve, described as the path of the motion of the system (see Fig. 1.5). By motion of the system, we actually mean, motion of the system point along this path in configuration space. It is necessary to mention here that the configuration space has no necessary connection with the cartesian three dimensional space, just as the generalized coordinates are not necessarily position coordinates.

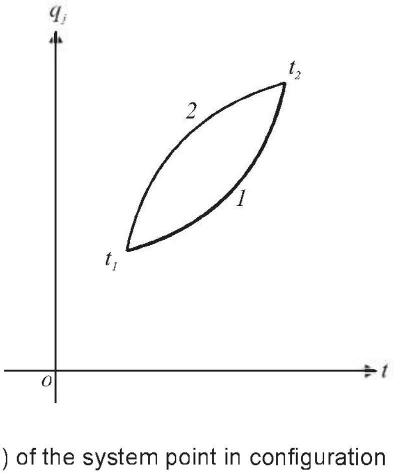

Fig. 1.5 Actual path (1) of the system point in configuration space along which I is extremum. 2 corresponds to varied path obtained by virtual variation

Let us now state the integral or Hamilton's principle, according to which 'The motion of the conservative system from time t_1 to t_2 is such that the line integral

$$I = \int_{t_1}^{t_2} L\left(q_1, q_2 \ldots\ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots\ldots.\dot{q}_s, t\right) dt \tag{48}$$

is extremum for the correct path of motion.' The function $L = T\text{-}V$ is of course called the Lagrangian of the system concerned and the quantity I is called the action or action integral.

That is, out of all possible paths by which the system point could travel from its position at t_1 to its position at t_2, it will actually travel along that path for which the value of the above integral is an extremum or in other words, we can say, that the motion is such that the variation of the line integral I for fixed t_1 to t_2, is zero, that is

$$\delta I = \delta \int_{t_1}^{t_2} L\left(q_1, q_2 \ldots\ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots\ldots.\dot{q}_s, t\right) dt = 0 \tag{49a}$$

Since there are no variations at the ends or since the end points are fixed,

$$\delta q_j(t_1) = \delta q_j(t_2) = 0, \qquad j = 1, 2, \ldots\ldots., s \tag{49b}$$

From (49), effecting the variation

$$\int_{t_1}^{t_2} \sum_j \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt = 0 \tag{50}$$

as there is no variation in time or time is not changing when we go from one curve to the other, that is, $\delta t = 0$.

Since d and δ are independent variations as d is variation in the same curve whereas with the help of δ one can go from one curve to the other without any change in time, therefore, we can interchange them. As a result

$$\delta \dot{q}_j = \delta\, (dq_j/dt) = d/dt\, (\delta q_j)$$

In the light of the above, equ. (50) takes the form

$$\int_{t_1}^{t_2} \sum_j \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \left(\delta q_j \right) \right) dt = 0 \tag{51}$$

On integrating 2nd term of the above integral by parts, one gets

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} (\delta q_j)\, dt = \frac{\partial L}{\partial \dot{q}_j} \left(\delta q_j \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \delta q_j\, dt$$

In view of the conditions (49b), the first term on the *r.h.s.* of the above equation is zero. As a result equ. (51) reduces to

$$\int_{t_1}^{t_2} \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j dt = 0$$

Since all δq_j's are independent of each other, the value of the above integral would be zero only if the coefficients of δq_j separately vanish. Thus we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, 2, \ldots, s$$

These are the required differential equations, called Lagrange's equations of motion.

1.7 SUPERIORITY OF LAGRANGIAN MECHANICS OVER NEWTONIAN MECHANICS

In Newtonian mechanics for setting up the equation of motion, that is, equ. (2), a complete knowledge of the forces which are vector quantities, is required whereas in Lagrangian mechanics, one has to deal with only two scalar functions T and V which drastically simplifies the problem. Let us now describe the procedure for solving problems in mechanics using Lagrangian formulation. To do so, one has to write T and V in generalized coordinates, form L from there and then substitute in equ. (28) to obtain the equations of motion. The required transformation of T and V from Cartesian coordinates to generalized coordinates is obtained by applying the transformation equations (11) and (20c),

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots\ldots\ldots, q_s, t), i = 1, 2, \ldots\ldots\ldots N \qquad (11)$$

As a result T is given by

$$T = \sum_i \tfrac{1}{2} m_i \upsilon_i^2 = \sum_i^N \tfrac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$$

$$= \sum_{i=1}^N \tfrac{1}{2} m_i \left[\left(\sum_{j=1}^s \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\right)\cdot\left(\sum_{k=1}^s \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}\right)\right]$$

$$= \sum_i \tfrac{1}{2} m_i \left[\sum_{j,k} \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_j\dot{q}_k + 2\sum_j \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial t}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\cdot\frac{\partial \vec{r}_i}{\partial t}\right]$$

$$= a + \sum_j a_j \dot{q}_j + \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k \qquad (52a)$$

where

$$a = \sum_{i=1}^N \tfrac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial t}\cdot\frac{\partial \vec{r}_i}{\partial t} = \sum_{i=1}^N \tfrac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2$$

$$a_j = \sum_{i=1}^N m_i \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial t}$$

$$a_{jk} = \sum_{i=1}^N \tfrac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial q_k} \qquad (52b)$$

are definite functions of $\vec{r}$'s and t and hence of $q_1, q_2, \ldots\ldots q_s, t$

Thus the kinetic energy T of a system can always be written as the sum of three homogenous functions of the generalized velocities,

$$T_0 + T_1 + T_2 \tag{52c}$$

where T_0 is independent of the generalized velocities, T_1 is linear in the velocities and T_2 is quadratic in the velocities.

If the transformation equations do not contain time explicitly, as may occur when the constraints are independent of time (or are scleronomous), then the first and second terms in equ. (52a) would vanish as the partial derivatives w.r.t. time vanish. The only last term in equ. (52c) would be non-vanishing in that case and as a result T would be a homogeneous quadratic function of generalized velocities $\dot{q}_j$:

$$T = \tfrac{1}{2} \sum_{j=1}^{s} \sum_{k=1}^{s} a_{jk}\, \dot{q}_j\, \dot{q}_k \tag{52d}$$

Let us consider a simple example of this procedure:

Example 2. Atwood's machine - Consider a system of two masses m_1 and m_2 tied by a light (massless) inextensible string of length l. The masses are hanging over a pulley (frictionless) as shown in Fig. 1.6. Obtain Lagrange's equation of motion for the system.

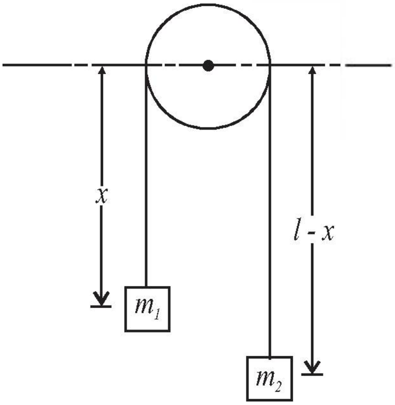

Fig. 1.6 Atwood's machine

It is an example of a conservative system with holonomic scleronomous constraint as the pulley is frictionless. There is only one independent coordinate x in this case. The position of the other mass being determined by the constraint that the length of string between m_1 and m_2 is l. The potential energy is

$$V = -m_1 g x - m_2 g(l-x)$$

while the kinetic energy is

$$T = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \left(\dot{\overline{l-x}}\right)^2 = \tfrac{1}{2}(m_1 + m_2)\dot{x}^2$$

Therefore, the Lagrangian

$$L = T - V = \tfrac{1}{2}(m_1 + m_2)\ \dot{x}^2\ + m_1 g x + m_2 g\,(l-x)$$

Substituting above in the Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0,$$

one gets

$$(m_1 + m_2)\,\ddot{x} - (m_1 - m_2)g$$

or
$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2}\,g$$

which is a well known result obtained by Newtonian mechanics. Note that the forces of constraint, the tension in string in the present case appear no where in the Lagrangian formulation. At the same time it simply means that the method can not furnish the tension in the string.

PROBLEMS

1.1 A double pendulum vibrates in a vertical plane (see Fig. 1.7). If masses of the bobs are m_1 and m_2 and lengths of the strings are l_1 and l_2, respectively, obtain Lagrangian and Lagrange's equations of motion for this double pendulum.

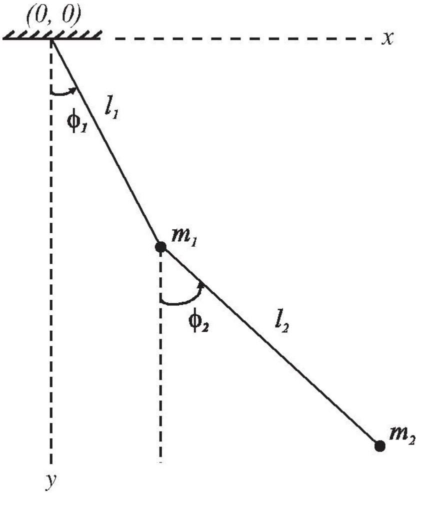

Fig. 1.7 Double pendulum

1.2 A particle of mass m moves on a smooth horizontal table under the action of an attached spring that passes through a hole and is attached at the other end to a spring of spring constant k (see Fig. 1.8). Obtain Lagrangian and Lagrange's equation of motion for the particle assuming that the point to which lower end of the spring is rigidly attached is so placed that when m is at the hole, the spring is unstretched.

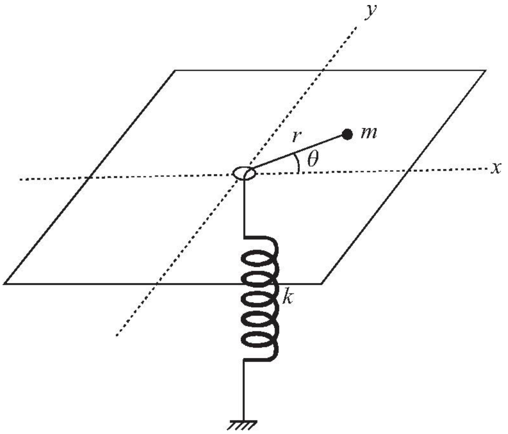

Fig. 1.8 Problem 1.2

1.3 Consider a simple pendiulum of mass m_2, with a mass m_1 at the point of support which can move on a horizontal line lying in the plane in which m_2 moves (see Fig. 1.9). Obtain Lagrangian of the system.

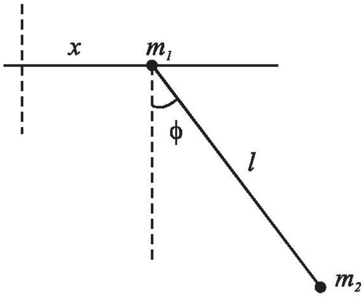

Fig. 1.9 Problem 1.3

1.4 Obtain Lagrangian and Lagrange's equation of motion for a spherical pendulum (essentially a simple pendulum whose bob moves about the point of suspension (see Fig. 1.10) on the surface of a sphere whose radius r is equal to the length of pendulum). Note that the bob moves in a circle in the case of a simple pendulum.

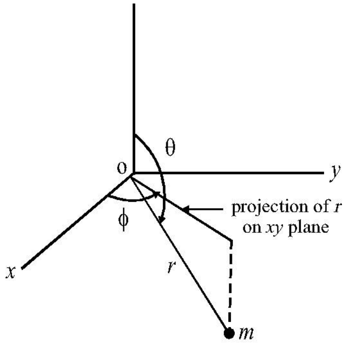

Fig. 1.10 Spherical pendulum

1.5 Obtain equation of motion for a particle of mass m falling vertically under the influence of gravity when frictional forces obtainable from a dissipation function ½ kv^2 are present. Integrate the equation to obtain the velocity as a function of time and show that the maximum possible velocity for a fall from rest is v = mg/k.

1.6 A simple pendulum of mass m and length l has a support point which moves uniformly with constant angular velocity ω on a vertical circle of radius a (see Fig. 1.12). Determine the motion of the pendulum bob.

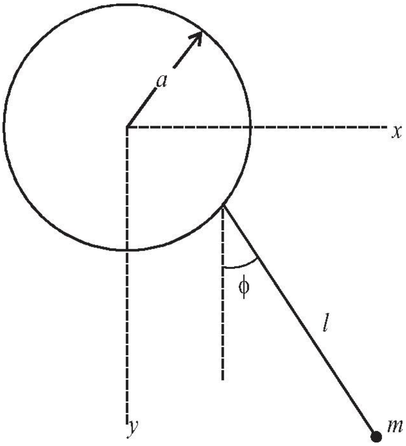

Fig. 1.12 Problem 1.6

1.7 A bead slides on uniformly rotating straight wire about an axis perpendicular to the wire in a force free space. Obtain Lagrangian and there from the equation of motion of the bead.

1.8 A compound (or physical) pendulum consists of a rigid body suspended in the vertical plane at a point other than the centre of gravity and swinging about the horizontal axis passing through the point (see Fig. 1.13). Compute the period for small oscillations.

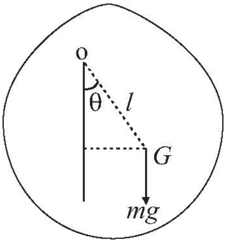

Fig. 1.13 Compound pendulum

1.9 A particle of mass m is revolving about some mass M. Obtain Lagrangian of the system and the Lagrange's equation of motion.

1.10 Consider an electrical circuit containing an inductance, a capacitance, and a switch. If the capacitor is charged to Q and the switch is, then closed, oscillations would be established in the circuit. Find the differential equation giving the charge on the capacitor as a function of time (see Fig. 1.15).

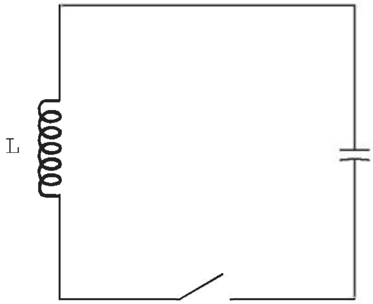

Fig. 1.15 An oscillating electrical circuit

1.11 A particle slides on the inside of a smooth paraboloid of revolution (See Fig. 1.16) whose axis of symmetry is vertical. Using the distance r of particle from the symmetry axis and its azimuth θ as the generalized coordinates find (a) Lagrange's equations of motion and (b) the angular momentum necessary for particle to move in the horizontal circle.

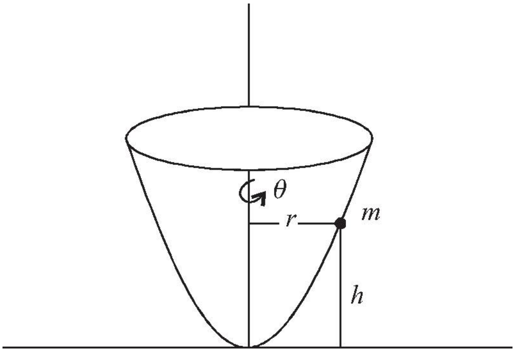

Fig. 1.16 Particle sliding inside a smooth paraboloid of revolution

SOLUTIONS

1.1 Let (x_1, y_1) and (x_2, y_2) represent the position coordinates of the bobs of masses m_1 and m_2, respectively (with origin at the point of support and the y-axis vertically downwards). From Fig. 1.7,

$$x_1 = l_1 \sin\phi_1,\ y_1 = l_1 \cos\phi_1$$

and $$x_2 = l_1 \sin\phi_1 + l_2 \sin\phi_2,\ y_2 = l_1 \cos\phi_1 + l_2 \cos\phi_2$$

Therefore, we have

For bob 1, the kinetic energy,

$$T_1 = ½\, m_1(\dot{x}_1^2 + \dot{y}_1^2) = ½\, m_1\, l_1^2 \dot{\phi}_1^2$$

and the potential energy,

$$V_1 = -\,m_1\, g l_1 \cos\phi_1$$

For bob 2, the kinetic energy,

$$T_2 = ½\, m_2\, (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= ½\, m_2 \left[\dot{\overline{\{l_1 \sin\phi_1 + l_2 \sin\phi_2\}}}^{\,2} + \dot{\overline{\{l_1 \cos\phi_1 + l_2 \cos\phi_2\}}}^{\,2} \right]$$

$$= ½\, m_2\, [(l_1 \cos\phi_1\dot{\phi}_1 + l_2 \cos\phi_2\dot{\phi}_2)^2 + (-l_1 \sin\phi_1\dot{\phi}_1 - l_2 \sin\phi_2\dot{\phi}_2)^2$$

$$= ½\, m_2\, [(l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2 l_1 l_2\, \dot{\phi}_1\dot{\phi}_2\, \cos(\phi_1 - \phi_2))]$$

and the potential energy.

$$V_2 = -m_2\, g\, y_2 = -\, m^2\, g\, (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$

As a result, the Lagrangian is

$$L = T - V = ½\, m_1\, l_1^2\dot{\phi}_1^2 + ½\, m_2\, [\, l_1^2\dot{\phi}_1^2 + l_2^2\dot{\phi}_2^2 + 2\, l_1 l_2 \dot{\phi}_1\, \dot{\phi}_2 \cos(\phi_1 - \phi_2)]$$

$$+ m_1 g\, l_1 \cos\phi_1 + m_2\, g\, (l_1 \cos\phi_1 + l_2 \cos\phi_2)$$

$$= ½\, (m_1 + m_2)\, l_1^2\dot{\phi}_1^2 + ½\, m_2\, l_2^2\dot{\phi}_2^2 + m_2\, l_1\, l_2\, \dot{\phi}_1\, \dot{\phi}_2\, \cos(\phi_1 - \phi_2)$$

$$+ (m_1 + m_2)\, g\, l_1 \cos\phi_1 + m_2\, g\, l_2 \cos\phi_2 \qquad (53)$$

The Lagrange's equations in the present case are given as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_1} - \frac{\partial L}{\partial \phi_1} = 0 \quad \text{and} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_2} - \frac{\partial L}{\partial \phi_2} = 0 \qquad (54)$$

From equ. (53), one obtains

$$\frac{\partial L}{\partial \phi_1} = -\, m_2\, l_1\, l_2 \dot{\phi}_1\, \dot{\phi}_2\, \sin(\phi_1 - \phi_2) - (m_1 + m_2)\, g\, l_1 \sin\phi_1,$$

$$\frac{\partial L}{\partial \dot{\phi}_1} = (m_1 + m_2)\, g\, l_1^2\, \dot{\phi}_1 + m_2\, l_1\, l_2 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_1} = (m_1 + m_2)\, l_1^2 \ddot{\phi}_1 + m_2\, l_1\, l_2 \ddot{\phi}_2\, \cos(\phi_1 - \phi_2),$$

$$\frac{\partial L}{\partial \phi_2} = m_2\, l_1\, l_2\, \dot{\phi}_1\, \dot{\phi}_2 \sin(\phi_1 - \phi_2) - m_2\, g\, l_2 \sin\phi_2,$$

$$\frac{\partial L}{\partial \dot{\phi}_2} = m_2\, l_2^2\dot{\phi}_2 + m_2\, l_1\, l_2\, \dot{\phi}_1\, \cos(\phi_1 - \phi_2),$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_2} = m_1\, l_2^2\ddot{\phi}_2 + m_2\, l_1\, l_2\ddot{\phi}_1 \cos(\phi_1 - \phi_2)$$

Substituting above into equs. (54), the Lagrange's equations become

$$(m_1 + m_2)\, l_1^2\ddot{\phi}_1 + m_2\, l_1\, l_2\ddot{\phi}_2 \cos(\phi_1 - \phi_2) + m_2\, l_1\, l_2\dot{\phi}_1\, \dot{\phi}_2 \sin(\phi_1 - \phi_2)$$
$$= -(m_1 + m_2)\, g\, l_1 \sin\phi_1 \qquad (55)$$

$$m_2\, l_2^2\ddot{\phi}_2 + m_2\, l_1\, l_2\ddot{\phi}_1 \cos(\phi_1 - \phi_2) - m_2\, l_1\, l_2\dot{\phi}_1\, \dot{\phi}_2 \sin(\phi_1 - \phi_2)$$
$$= -\, m_2\, g\, l_2 \sin\phi_2 \qquad (56)$$

1.2 The kinetic energy in this case is

$$T = \tfrac{1}{2}\, m(\dot{r}^2 + r^2\dot{\theta}^2)$$

and the potential energy is

$$V = \tfrac{1}{2}\, k\, r^2$$

Therefore, the Lagrangian

$$L = T - V = \tfrac{1}{2}\, m\,(\dot{r}^2 + r^2\dot{\theta}^2) - \tfrac{1}{2}\, kr^2$$

Now using the Lagrange's equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

in the present case, one gets

$$m\ddot{r} - mr\dot{\theta}^2 = -kr \qquad (57)$$

and $$d/dt(mr^2\dot{\theta}) = 0 \qquad (58)$$

From equ. (58), one infers that

$$mr^2\dot{\theta} = p_\theta = \text{angular momentum} = \text{constant}, \qquad (59)$$

Putting the value of $\dot{\theta}$ from equ. (59) into equ. (57), one gets

$$m\ddot{r} - p_\theta^2/mr^3 = -kr$$

which could be solved for r

In terms of the cartesian coordinates, the kinetic energy

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$$

and the potential energy

$$V = \tfrac{1}{2}\, kr^2 = \tfrac{1}{2}\, k\,(x^2 + y^2)$$

From above, the Lagrange's equations of motion are

$$m\ddot{x} = -kx \text{ and } m\ddot{y} = -ky$$

Therefore, the motion is composed of two simple harmonic motions at right angles with same frequency $\omega = (k/m)^{1/2}$

That is,

$$x = A \sin \omega t,$$

and $$y = B \text{ sing } (\omega t + 90^\circ) = B \cos \omega t$$

As a result the path of the particle is ellipse.

1.3 For mass m_1, the kinetic energy

$$T_1 = ½ m_1 \dot{x}^2$$

and the potential energy

$$V_1 = 0$$

For mass m_2, the kinetic energy

$$T_2 = ½ m_2 \left[\left(\overset{\cdot}{\overline{x + l\sin\phi}}\right)^2 + \left(\overset{\cdot}{\overline{l\cos\phi}}\right)^2\right]$$

$$= ½\ m_2 \left(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\cos\phi\,\dot{\phi}\right)$$

and the potential energy,

$$V_2 = -m_2\, g\, l \cos \phi$$

Therefore, Lagrangian of the system

$$L = T - V = ½\, m_1 \dot{x}^2 + ½\, m_2\, (\dot{x}^2 + l^2\dot{\phi}^2 + 2\ \dot{x}\, l \cos\phi\,\dot{\phi}) + m_2\, g\, l \cos\phi$$

$$= ½\, (m_1 + m_2)\ \dot{x}^2 + ½\, m_2\, l^2\dot{\phi}^2 + m_2\ \dot{x}\, l \cos\phi\,\dot{\phi} + m_2\, g\, l\cos\phi$$

$$= ½\, (m_1 + m_2)\ \dot{x}^2 + ½\, m_2\, (l^2\dot{\phi}^2 + 2\, l\ \dot{x}\ \dot{\phi}\ \cos\phi) + m_2\, g\, l\cos\phi$$

1.4 The spherical polar coordinates are most suitable to locate the position of the bob. Since r (length of the pendulum) is constant, only θ and ϕ need to be specified.

The kinetic energy of the bob is

$$T = ½\, m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right), x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi, z = r\cos\theta$$

$$= ½\, m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

$$= ½\, m\, r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right),$$

as r, the length of pendulum is fixed.

The potential energy due to gravity, relative to the horizontal plane $z = 0$, is

$$V = -m\, g\, r\cos\theta$$

As a result, the Lagrangian is given by

$$L = T - V = ½ m\, r^2\dot{\theta}^2 + ½ mr^2\sin^2\theta\dot{\phi}^2 + mgr\cos\theta,$$

hence the Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0, \quad \text{and} \quad \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} = 0$$

take the forms

$$mr^2\ddot{\theta} - mr^2 \sin\theta \cos\theta\, \dot{\phi}^2\ \ \dot{\phi} + mgr \sin\theta = 0 \tag{59}$$

and

$$d/dt\,(mr^2 \sin^2\theta\, \dot{\phi}) = 0 \tag{60}$$

From equ. (60), one infers that

$$mr^2 \sin^2\theta\, \dot{\phi} = p_\phi = \text{constant} \tag{61}$$

Substituting the value of $\dot{\phi}$ in terms of p_ϕ into equ. (59), one gets equation of motion in θ.

$$mr^2\ddot{\theta} = -mgr \sin\theta + p_\phi^2 \cos\theta/mr^2 \sin^3\theta \tag{62}$$

To see the limiting case, let us assume that the pendulum moves in the $\phi = 0$ plane. The above equation then reduces to

$$\ddot{\theta} = -g/r \sin\theta$$

which on further assuming that θ is small, that is, $\sin\theta = \theta$ reduces to

$$\ddot{\theta} = -g/r\ \theta,$$

the well known equation of motion for a simple pendulum.

1.5 Let us assume that the particle is at distance of y from the ground at time t (see Fig. 1.11). Its kinetic energy, therefore would be

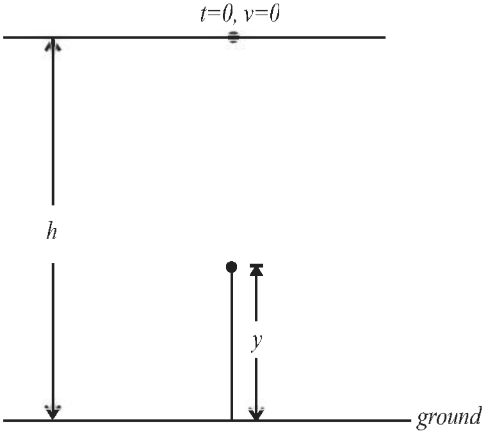

Fig. 1.11 Problem 1.5

$$T = \tfrac{1}{2} m\dot{y}^2$$

and the potential energy

$$V = mgy$$

As a result, the Lagrangian is

$$L = T - V = \tfrac{1}{2} m\dot{y}^2 - mgy$$

The dissipation function is given by

$$R = \tfrac{1}{2}\, k\, v^2 = \tfrac{1}{2}\, k\dot{y}^2$$

Substituting the values of L and R in the Lagrange's equation in the presence of the dissipation function,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \frac{\partial R}{\partial \dot{y}} = 0,$$

one gets

$$m\ddot{y} + k\dot{y} = mg \tag{63}$$

On integrating the above equation of motion, one gets

$$m\dot{y} + ky = mgt + C$$

where C is the constant of integration.

Using the boundary conditions, that is,

at $\quad t = 0, v = \dot{y} = 0$ and $y = h$,

$$C = 0$$

This amounts to

$$m\dot{y} + ky = mgt + kh \tag{64}$$

which gives velocity of the particle as a function of time.

For finding the maximum velocity for a fall from rest, we have to put $dv/dt = d\dot{y}/dt = 0$ in the above equation. Doing so, we get

$$\dot{y}_{\max} = v_{\max} = \mathrm{mg/k}$$

1.6 Let m be at point (x, y). Therefore

$$x = a\cos\omega t + l\sin\phi, \text{ and } y = -a\sin\omega t + l\cos\phi$$

where ω is the angular velocity of the support point. It may be noted that the constraint is dependent on time in this case.

The kinetic energy of masses,

$$T = \tfrac{1}{2}\, m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$= \tfrac{1}{2}\, m\, l^2\dot{\phi}^2 + \tfrac{1}{2}\, m\, a^2\,\omega^2 - m\, a\, l\,\omega\,\sin(\omega t - \phi)\,\dot{\phi}$$

and the potential energy,

$$V = -mg\, l\cos\phi$$

As a result, the Lagrangian is

$$L = T - V = \tfrac{1}{2}\, ml^2\dot{\phi}^2 + \tfrac{1}{2}\, ma^2\,\omega^2 - m\, al\omega\,\sin(\omega t - \phi)\,\dot{\phi} + mgl\cos\phi$$

and hence the Lagrange's equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0,$$

in the present case takes the form

$$ml^2\ddot{\phi} - m\,a\,l\,\omega\cos(\omega t - \phi)\dot{\phi} = m\,a\,l\,\omega^2\cos(\omega t - \phi) + m\,g\,l\sin\phi$$

or $$l\ddot{\phi} - a\,\omega\cos(\omega t - \phi)\dot{\phi} = a\omega^2\cos(\omega t - \phi) + g\sin\phi$$

1.7 The equations of transformation in this case are

$$x = r\cos\theta = r\cos\omega t$$

and $$y = r\sin\theta = r\sin\omega t$$

where ω is the angular velocity of rotation of the wire. Note that this is an example of the constraint being time dependent.

The velocities are, therefore, given as

$$\dot{x} = \dot{r}\cos\omega t - r\sin\omega t\,\omega,$$
$$\dot{y} = \dot{r}\sin\omega t + r\cos\omega t\,\omega$$

The kinetic energy T (here the same as L) as a result reduces to

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right)$$

Thus we find that T is not a homogenous quadratic function of the generalized velocities. The Lagrange's equation of motion in the present case

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} - \frac{\partial T}{\partial r} = 0,$$

takes the form,

$$m\ddot{r} = mr\,\omega^2$$

or $$\ddot{r} = r\,\omega^2$$

the result which is well known indicating that the bead moves outward because of the centripetal acceleration. The method, of course, is unable to furnish the force of constraint keeping the bead on the wire.

1.8 Let O be the point of suspension through which the axis of rotation passes, G the centre of gravity and θ the angle (of rotation) through which the body is deflected (see Fig. 1.13). Therefore, the kinetic energy of the body is

$$T = \tfrac{1}{2}I\dot{\theta}^2$$

where I is the moment of inertia of the body about the axis of rotation.

The potential energy of the body relative to the horizontal plane through O is

$$V = -mg\,l\cos\theta$$

As a result, the Lagrangian

$$L = T - V = \tfrac{1}{2}I\dot{\theta}^2 + mgl\cos\theta$$

and therefore, the Lagrange's equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

in this case, takes the form

$$\ddot{\theta} + mg\, l/I \;\sin\theta = 0 \tag{65}$$

For small oscillations, $\sin\theta \approx \theta$ and the above equation takes the form

$$\ddot{\theta} = -\,mg\, l/I \;\theta, \tag{66}$$

a well known equation of motion for a simple pendulum. The time period is

$$T = 2\pi/\omega = 2\pi\sqrt{I/mgl}$$

Before we close the discussion on the above problem, it is necessary to mention here that Kater's and bar pendulums are examples of compound pendulum.

1.9 Let r be the distance between two masses (see Fig. 1.14). Therefore, the force of attraction between them is

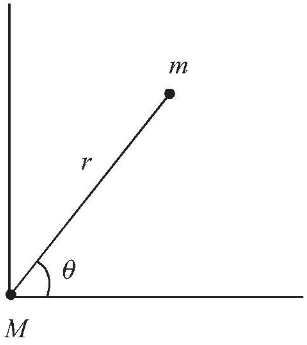

Fig. 1.13 Problem 1.9

$$F = -GM\,m/r^2,$$

where the minus sign is introduced because of the fact that the (+)ve direction is taken along the increasing r.

The potential energy between the two masses,

$$V = \int_{\infty}^{r} -F\,dr = -GM\,m/r$$

The kinetic energy of mass m is given by

$$T = ½\, m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Therefore, the Lagrangian

$$L = T - V = ½\, m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + GM\,m/r$$

In the present case, there are two generalized coordinates

$$q_1 = r, \quad \text{and} \quad q_2 = \theta$$

and the corresponding two Lagrange's equations of motion would be given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

or $$d/dt\,(m\dot{r}) = mr\dot{\theta}^2 + GMm/r^2 = 0$$

or $$m\left(\ddot{r} - r\dot{\theta}^2\right) - GMm/r^2 \tag{67}$$

the radial equation of motion where $\ddot{r} - r\dot{\theta}^2$ is the radial acceleration and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

or $$d/dt\,(mr^2\,\dot{\theta}) = 0 \tag{68}$$

which amounts to be conservation of angular momentum $p_\theta = mr^2\dot{\theta}$

1.10 The energy stored in the magnetic field of the inductance £(in analogous to the kinetic energy $\frac{1}{2}mv^2$, where v is analogous to the current dQ/dt, and m is analogous to inductance £) is $\frac{1}{2}\,\pounds\dot{Q}^2$, where Q is the instantaneous charge on the capacitor. The energy stored in the capacitor C (like $\frac{1}{2}\,kx^2$, the potential energy for the spring. k is a analogous to $1/C$ and x the displacement is analogous to Q) is $\frac{1}{2}\,(1/C)\,Q^2$. As a result, the Lagrangian is

$$L = T - V = \tfrac{1}{2}\,\pounds\dot{Q}^2 - \tfrac{1}{2}(1/C)\,Q^2$$

and hence the Lagrange's equation of motion choosing Q as the generalized coordinate is

$$\frac{d}{dt}\frac{\partial L}{\partial} \quad \frac{\partial L}{\partial Q} = 0 = \ddot{Q} + 1/\pounds C\,Q$$

or $$\ddot{Q} + \omega^2 Q = 0, \omega = \sqrt{1/\pounds C},$$

which is nothing but a familiar oscillator equation from electrical resonance theory.

1.11 The equation of constraint is given by (see Fig. 1.16)

$$h = ar^2 \tag{69}$$

where a is a constant.

The potential and kinetic energies of the particle of mass m are given by

$$V = mgh = mg\,ar^2,$$

$$T = \tfrac{1}{2}\,m\dot{r}^2 + \tfrac{1}{2}\,m\dot{h}^2 + \tfrac{1}{2}\,mr^2\dot{\theta}^2$$

$$= \tfrac{1}{2}\,m\dot{r}^2 + \tfrac{1}{2}\,m\,(2ar\dot{r})^2 + \tfrac{1}{2}\,mr^2\dot{\theta}^2$$

$$= \tfrac{1}{2}\,m\dot{r}^2 + 2\,ma^2r^2\dot{r}^2 + \tfrac{1}{2}\,mr^2\dot{\theta}^2$$

where use of equ. (69) has been made.

As a result, the Lagrangian

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2} m\dot{r}^2 + 2\, ma^2r^2\dot{r}^2 + \tfrac{1}{2} mr^2\dot{\theta}^2 - mgar^2 \end{aligned}$$

and hence the Lagrange's equations of motion would be given by

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= \frac{d}{dt}(m\dot{r} + 4\, ma^2r^2\dot{r}) - 4\, ma^2r\dot{r}^2 \\ &\quad - mr\dot{\theta}^2 + 2\, mgar = 0 \end{aligned} \tag{70}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{71}$$

which implies that

$$mr^2\dot{\theta} = \text{angular momentum } p_\theta = \text{constant}$$

Further the motion in the horizontal circle implies that

$$r = \text{constant},$$

which further implies that

$$\dot{r} = \ddot{r} = 0$$

Substitution of the above condition into equ. (70) leads to

$$\dot{\theta}^2 = 2\, ga$$

or

$$\dot{\theta} = (2\, ga)^{1/2}$$

As a result the angular momentum

$$p_\theta = mr^2\dot{\theta} = mr^2\,(2\, ga)^{1/2}$$

CHAPTER 2

Some Aspects of the Calculus of Variations

2.1 EULER-LAGRANGE'S EQUATION

The major problem of the calculus of variations is to find the path (curve) for which some given line integral is an extremum. Though the development of this technique was begun by Newton, but Bernoulli, Legendre, Euler, Lagrange, Hamilton and Jacobi made important contributions.

Let us now try to find a path (curve) y = y(x) between the limits x_1 and x_2 such that the line integral.

$$I = \int_{x_1}^{x_2} f(y, y', x)\, dx \tag{1}$$

of some function $f(y, y', x)$ where $y' = dy/dx$, is an extremum relative to paths differing infinitesimally from the correct function $y(x)$.

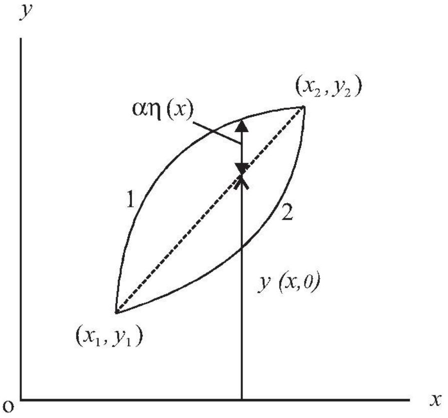

Fig. 2.1 Varied paths having the same initial and final points

In Fig. 2.1 dotted path is the one which gives extremum value to I. Let us label all the neighbouring paths $y(x)$ (say curves 1 and 2) under the examination with different values of a parameter α, such that for some value of α, say $\alpha = 0$, the path would coincide with the path (correct path to be precise) giving an extremum for the line integral. y in that case would be function of x and α. This simply means that a possible set of varied paths is given by

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x) \tag{2}$$

where $\eta(x)$ is some function of x, which vanishes at $x = x_1$ and $x = x_2$ where all paths meet. That is

$$\eta(x_1) = \eta(x_2) = 0 \tag{3}$$

and $y(x,0)$ of course represents the correct path.

The above arguments imply that I is a function of α and hence we can write equ. (1) as

$$I(\alpha) = \int_{x_1}^{x_2} f[y(x, \alpha), y'(x, \alpha), x]dx \tag{4}$$

and the condition for extremum would of course be

$$\left.\frac{\partial I}{\partial \alpha}\right|_{\alpha=0} = 0 \tag{5}$$

Differentiating (4) under the integral sign by the usual method, one gets

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} \right) dx \tag{6}$$

as the term $\frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}$ vanishes because of the fact that for the same value of x, α is different (variable x is independent of α) or $\frac{\partial x}{\partial \alpha} = 0$,

From equ. (2), we see that

$$\frac{\partial y(x,\alpha)}{\partial \alpha} = \eta(x), \frac{\partial y'(x,\alpha)}{\partial \alpha} = \frac{d}{dx}\frac{\partial y}{\partial \alpha} = \frac{d\eta(x)}{dx}$$

where we have interchanged the differentials with respect to x and α as they are independent. As a result, equ. (6) takes the form

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta(x)}{dx} \right) dx \tag{7}$$

Integrating the second term on the *r.h.s.* in the above equation by parts gives

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d\eta(x)}{dx}dx = \frac{\partial f}{\partial y'}\eta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)dx$$

The first term on the *r.h.s.* of the above equation vanishes as a result of equ. (3), and equ. (7), therefore reduces to

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] \eta(x)dx \tag{8}$$

Note that the functions y and y' *w.r.t.* which the derivatives of f are taken, are functions of α. When $\alpha = 0$, we have $y(x, \alpha) = y(x)$, and the dependence on α disappears. We required that $\left.\frac{\partial I}{\partial \alpha}\right|_{\alpha=0} = 0$, and since $\eta(x)$ is an arbitrary function except at the end points where it vanishes, that is $\eta(x_1) = \eta(x_2) = 0$, the integrand of equ. (8) must vanish for $\alpha = 0$. Therefore, we have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \tag{9}$$

called Euler - Lagrange's equation which represents a necessary condition for I to be extremum.

2.2 GENERALIZATION FOR SEVERAL VARIABLES

The above problem of the calculus of variations can be easily generalized to the case where f is a function of many independent variables y_i, and their derivatives y_i', considering all these quantities to be functions of parametric variable x. That is,

$$f = f[y_i(x),\ y_i'(x), x], \qquad i = 1, 2, \ldots\ldots\ldots, s \tag{10}$$

In this case, we write equ. (2) as

$$y_i(x, \alpha) = y_i(x, 0) + \alpha\eta_i(x) \tag{11}$$

where the η_i's are independent of one another except for the fact that they all vanish at the end points.

Following the same procedure as above, we get a set of differential equations

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0, \qquad i = 1, 2, \ldots\ldots\ldots, s \tag{12}$$

called Euler-Lagrange's equations.

Note that the integral involved in Hamilton's principle,

$$I = \int_{t_1}^{t_2} L(q_i,\ \dot{q}_i, t)dt$$

resembles the present line integral,

$$\int_{x_1}^{x_2} f(y_i,\ y_i',\ x)dx$$

with the transformations

$$x \to t, \quad y_i = q_i,$$

that is,

$$f(y_i,\ y_i', x) \to L(q_i,\ \dot{q}_i, t)$$

In deriving equ. (12) it was assumed that y_i variables are independent. The corresponding condition in connection with Hamilton's principle is that the generalized coordinates q_i be independent, which requires that the constraints be holonomic. The Euler - Lagrange's equations corresponding to the integral I then become the Lagrange's equations of motion.

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \qquad i = 1, 2, \ldots\ldots\ldots\ldots, s \tag{12}$$

and we have accomplished our aim to show that Lagrange's equations of motion follow from Hamilton's principle for conservative systems with holonomic constraints.

2.3 LAGRANGE'S UNDETERMINED MULTIPLIERS

Consider a function $f = f(x, y, z)$ of three independent variables, subject to certain constraints. The function f has an extremum value when

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0, \tag{13}$$

The necessary and sufficient condition for the satisfaction of the above equation is

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \tag{14}$$

Quite often the independent variables are subject to constraint or in other words they are no longer independent. In that case one can not obtain equ. (14). In principle, it is possible to use each constraint equation to eliminate a dependent variable and work with a new set of independent variables. In many cases, it is undesirable or inconvenient to eliminate the dependent variable. Under these circumstances, a technique known as Lagrange's multiplier is used.

Let the equation of constraint be

$$g(x, y, z) = 0 \tag{15}$$

from which one can write

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz = 0 \tag{16}$$

Because of the condition of constraint (15), the condition (14) is no longer satisfied since there are only two independent variables now(as there is only one equation of constraint and hence only one dependent variable can be eliminated). If we take x and y to be independent variables, then dz is no longer arbitrary. We can, however, add equs. (13) and (16) after multiplying the later by λ to write

$$df + \lambda dg = \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z}\right)dz = 0 \tag{17}$$

where the Lagrange's multiplier λ is chosen such that

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z} = 0, \quad \frac{\partial g}{\partial z} \neq 0 \tag{18}$$

With this condition equ. (17) becomes

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}\right)dy=0 \tag{19}$$

Since x and y are independent, to fulfill the above requirement, their coefficients must vanish. Therefore,

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=0 \tag{20}$$

Thus, when equs. (18) and (20) are satisfied, $df = 0$ or $f(x, y, z)$ is an extremum. Note that there are four unknowns x, y, z and λ in the problem and three equations (18), (20) and (15) for their solution. We need to find only x, y and z in order to solve the problem and λ is usually not determined. For this very reason, λ is called Lagrange's undetermined multiplier.

2.4 NONHOLONOMIC SYSTEMS AND FORCES OF CONSTRAINT

As mentioned earlier, the constraints which can be expressed as relations between the coordinates, are holonomic. If the system is subject to only such constraints, then a set of generalized coordinates can always be found, in terms of which the problem can be formulated without explicit reference to the constraints. This situation is not true when the constraints acting on the system are non-holonomic. For example, suppose that we have a constraint relation in the form

$$a\dot{x}+b\dot{y}+c\dot{z}+e = 0 \tag{21}$$

where a, b, c, e are functions of x, y, z, t. In general, this equation can not be integrated, and therefore, the constraint is nonholonomic. If however,

$$a=\frac{\partial\psi}{\partial x}, b=\frac{\partial\psi}{\partial y}, c=\frac{\partial\psi}{\partial z}, e=\frac{\partial\psi}{\partial t}$$

where $\psi = \psi(x, y, z, t)$, then equ. (21) becomes

$$\frac{\partial\psi}{\partial x}\frac{dx}{dt}+\frac{\partial\psi}{\partial y}\frac{dy}{dt}+\frac{\partial\psi}{\partial z}\frac{dz}{dt}+\frac{\partial\psi}{\partial t}=0=\frac{d\psi}{dt}$$

which can be integrated to give

$$\psi(x, y, z, t) - \text{const.} = 0$$

and the constraint is revealed to be holonomic after all. From the above example, we infer that the constraints expressible in the form

$$\sum_{j=1}^{s}\frac{\partial\psi_k}{\partial q_j}dq_j+\frac{\partial\psi_k}{\partial t}dt = 0 \tag{22}$$

are reducible to holonomic form. The index k labels the constraint relations.

As is customary, we now imagine virtual displacements δq_j with time, of course, held fixed, so that the 2nd term in equ. (22) vanishes and we are left with

$$\sum_j \frac{\partial \psi_k}{\partial q_j} \delta q_j = 0$$

The displacements δq_j are connected through the above relations, that is, they are not independent of one another. It is for this very reason that we introduce Lagrange's multipliers λ_k (one for each constraint relation). The sole motivation for doing so lies in the fact that the λ_k's are essential to a scheme which renders the δq_j effectively independent. Thus

$$\lambda_k \sum_j \frac{\partial \psi_k}{\partial q_j} \delta q_j = 0 \tag{23}$$

must hold.

Adding above, Lagrange's equations assume the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_k \lambda_k \frac{\partial \psi_k}{\partial q_j} \tag{24}$$

Further noting that the generalized force Q_j can be broken down into a part associated with a potential and a nonconservative (or non-potential) part or force of constraint $\vec{f}_j$ that is

$$Q_j = -\frac{\partial V}{\partial q_j} + \vec{f}_j \tag{25}$$

After doing so Lagrange's equations would assume the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \vec{f}_j \tag{26}$$

By comparison of equ. (26) with equ. (24), we find

$$\vec{f}_j = \sum_k \lambda_k \frac{\partial \psi_k}{\partial q_j} \tag{27}$$

The form of the relations above means that we can identify such a force of constraint with $\sum_k \lambda_k \frac{\partial \psi_k}{\partial q_j}$, so that it is found automatically from the Lagrange's multiplier when one uses this approach.

Example 1. Consider a disk of mass m rolling without slipping down an inclined plane of inclination ϕ (see Fig. 2.2). Solve the problem and find the force exerted by the constraint.

The kinetic energy of the disk is given by

$$\begin{aligned} T &= T_{\text{trans}} + T_{\text{rot}} \\ &= \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} I\dot{\theta}^2 \\ &= \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{4} mR^2\dot{\theta}^2 \end{aligned}$$

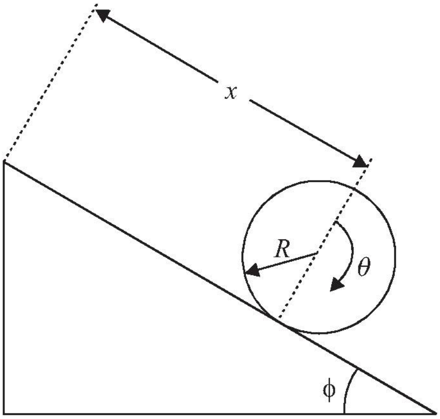

Fig. 2.2 Mass m rolling without slipping down an inclined plane

where $I = \frac{1}{2}$ m R^2 is the moment of inertia of the disk about its axis of rotation. The potential energy is

$$V = mg(l - x) \sin \phi$$

where l is the length of the inclined plane and V is taken to be zero at $x = l$.

Therefore, the Lagrangian is given by

$$L = T - V = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{4}\, mR^2\dot{\theta}^2 - mg(l - x) \sin\phi \tag{28}$$

The constraint in the present case is given by the requirement that the disk rolls without slipping, and hence the equation of constraint is

$$\psi(x, \theta) = R\theta - x = 0 \tag{29}$$

The Lagrange's equations of motion in x and θ, that is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \lambda\frac{\partial \psi}{\partial x},$$

and
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial \psi}{\partial \theta},$$

are
$$m\ddot{x} - mg \sin \phi + \lambda = 0, \tag{30}$$

$$\tfrac{1}{2}\, mR^2\ddot{\theta} - \lambda R = 0 \tag{31}$$

Now, from equ. (29) of constraint after differentiating the same twice, we get

$$R\ddot{\theta} = \ddot{x} \tag{32}$$

Eliminating θ between equs. (31) and (32), we get

$$\lambda = m\ddot{x}/2 \tag{33}$$

Substituting the above result in equ. (30) yields

$$\ddot{x} = 2/3\, g \sin \phi \tag{34}$$

Therefore, the Lagrange's multiplier (or the frictional force of constraint),

$$\lambda = 1/3\, mg \sin \phi \tag{35}$$

and hence from equ. (31),

$$\ddot{\theta} = 2/3Rg \sin \phi \tag{36}$$

It may be pointed out here that in the cases of cylinder rolling without slipping and a hoop rolling without slipping, the results would be same as obtained in equs. (35) and (36) except for the fact that the masses and the radii of the respective cases are to be used.

2.5 INTEGRALS OF MOTION (OR LAWS OF CONSERVATION)

Consider a system of N particles with s degrees of freedom. This would have s 2nd order differential equations. The solution of each equation would require two integrations resulting in 2 s constants of integration. These constants would be determined by the initial conditions, that is, by the initial values of the s q_j's and the s $\dot{q}_j$'s. We can express these 2 s arbitrary constants $C_1, C_2, \ldots. C_{2s}$ as functions of q and $\dot{q}$, and these functions would be integrals of motion (or constants of motion). These are relations of the type

$$f(q_1, \ldots\ldots\ldots, q_s, \dot{q}_1, \ldots\ldots\ldots\ldots, \dot{q}_s, t) = C \text{ (Const.)} \tag{37}$$

which are truly speaking first order differential equations.

Consider as an example a system of particles under the influence of forces derived from potentials dependent on position only, that is, conservative forces. Then

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \sum_i \tfrac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right) - 0$$

$$= m_i \dot{x}_i = p_{ix}$$

which is x - component of the linear momentum associated with the ith particle.

In analogous to this, one can define

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \qquad j = 1, 2, \ldots\ldots\ldots\ldots, s \tag{38}$$

as the generalized momentum associated with the generalized coordinate q_j. The terms canonical or conjugate momentum are often also used for p_j. Note that if q_j is not a Cartesian coordinate, p_j does not necessarily have the dimensions of a linear momentum. The dimensions of p_j are given by those of q_j : if q is the distance, then p is a linear momentum; if q is an angle, then p is an angular momentum etc.

Suppose the Lagrangian of a system does not contain a given coordinate q_j, although it may contain the corresponding velocity $\dot{q}_j$, then the coordinate is said to be cyclic or ignorable. The Lagrange's equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, ..., s$$

reduces, for a cyclic coordinate, to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - 0 = 0$$

or $$\frac{\partial p_j}{dt} = 0, \qquad j = 1, 2, ..., s$$

which implies that

$$p_j = \text{constant} \tag{39}$$

or in other words whenever a coordinate q_j does not appear explicitly in the Lagrangian of the system or is cyclic, the corresponding generalized momentum p_j is a constant of motion or in still other words, it is the first integral of motion. Thus equ. (39) constitutes a first integral of motion of the form (37) for the equation of motion.

Equ. (39) gives either the conservation of linear momentum or that of a angular momentum. It is not only valid in mechanics, but also for other physical systems. For example, we have a single charged particle in an electromagnetic field in which neither ϕ nor $\vec{A}$ depends on x. Then x appears no where in $L(= \frac{1}{2} mv^2 - e\phi + e\vec{v} \cdot \vec{A})$ and is therefore cyclic. The corresponding canonical momentums p_x must be conserved, that is

$$p_x = m\dot{x} + eA_x = \text{const.} \tag{40}$$

In this case it is not the mechanical linear momentum $m\dot{x}$ that is conserved but rather its sum with e A_x.

All integrals of motion, however are not of equal importance in mechanics. There are some integrals of motion whose constancy (or corresponding conservation laws) is of great important such as derived from the fundamental homogeneity of time (different instants would be equivalent) and space (if the body does not interact with the other bodies, its various positions in space would be mechanically equivalent) and from isotropy of space (if the body does not interact with the other bodies, its various orientations in the space would be mechanically equivalent). The quantities represented by such integrals of motion are said to be conserved and have an important common property of being additive, that is, there values for a system composed of several parts where interaction is negligible are equal to the sums of their values for the individual parts.

(i) Conservation of Energy

Let us first consider the conservation law resulting from the homogeneity of time (or symmetry with respect to the translation of the system along time). By virtual of this the Lagrangian of a closed system does not depend explicitly on time, that is

$$L = L(q_j, \dot{q}_j), \qquad j = 1, 2, \ldots.., s$$

Then the total time derivative of the Lagrangian is

$$\frac{dL}{dt} = \sum_{j=1}^{s}\left(\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right) \tag{41}$$

where the last term $\frac{\partial}{\partial} = 0$ as L does not depend explicitly on time or in other words as the translation of the system along time leaves L unchanged.

Now from Lagrange's equation of motion, we have

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}$$

Substituting above into equ. (41), one gets

$$\frac{dL}{dt} = \sum_{j=1}^{s}\dot{q}_j\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_{j=1}^{s}\frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$$

$$= \sum_{j=1}^{s}\frac{d}{dt}\left(\dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right)$$

or

$$\frac{d}{dt}\left(\sum_{j=1}^{s}\dot{q}_j\frac{\partial L}{\partial \dot{q}_j} - L\right) = 0 \tag{42}$$

Hence we conclude that the quantity inside the bracket on the *l.h.s.* of the above equation remains constant during the motion of a closed system (not interacting with others), that is, it is an integral of motion (or a constant of motion).

Let us assume that

$$\sum_{j=1}^{s}\dot{q}_j\frac{\partial L}{\partial \dot{q}_j} - L = H \text{ (a constant)} \tag{43}$$

We can show that H is nothing but total energy of the system. If the potential energy of the system is velocity and time independent, that is, $V = V(q_j)$, we can write

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T - V)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

and hence

$$H = \sum_{j=1}^{s}\dot{q}_j\frac{\partial T}{\partial \dot{q}_j} - L \tag{44}$$

From Euler's theorem for homogeneous function $f(q_j)$ of degree n, we have

$$\sum_{j=1}^{s} q_j\frac{\partial f}{\partial q_j} = nf$$

Since T is a homogenous quadratic function ($n = 2$) of generalized velocities (see Sec. 1.7) if constraints are independent of time or if the transformation equations do not contain time explicity, from Euler's theorem we have

$$\sum_{j=1}^{s} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T \tag{45}$$

Hence $$H = 2T - L = 2T - (T - V) = T + V, \tag{46}$$

which is total energy of the system. Thus the function H gives the total energy of the system (as sum of two terms, the kinetic energy which depends on the generalised velocities and the potential energy which depends only on the coordinates (positions) of the particles) and is called Hamiltonian of the system.

(ii) Conservation of Linear Momentum

A second conservation law follows from the homogeneity of space. By virtue of this homogeneity, the mechanical properties of a closed system are unchanged by any parallel displacement of the entire system in space. Let us, therefore, consider an infinitesimal displacement $\vec{\in}$ of the system and obtain the condition for Lagrangian to remain unchanged.

A parallel displacement is a transformation in which every particle in the system is moved by the same amount, that is, a radius vector $\vec{r}_i$ becoming $\vec{r}_i + \vec{\in}$. The change in Lagrangian resulting from the infinitesimal change in coordinates, the velocities of particles remaining fixed as the velocities are not affected by a shift in the origin, is

$$\delta L = \sum_{i=1}^{N} \frac{\partial L}{\partial \vec{r}_i} \cdot \delta \vec{r}_i = \vec{\in} \cdot \sum_{i=1}^{N} \frac{\partial L}{\partial \vec{r}_i}$$

where the summation is over the particles in the system and $\vec{\in}$ being same for all the particles.

Since $\vec{\in}$ is arbitrary, the condition $\delta L = 0$ is equivalent to

$$\sum_i \frac{\partial L}{\partial \vec{r}_i} = 0 \tag{47a}$$

The Lagrangian of the system is given by

$$L = \sum_{i=1}^{N} \tfrac{1}{2} m_i \upsilon_i^2 - V\left(\vec{r}_1, \vec{r}_1, ...\right),$$

therefore,

$$\frac{\partial L}{\partial \vec{r}_i} = -\frac{\partial V}{\partial \vec{r}_i} = \vec{F}_i \tag{47b}$$

As a result, the condition (47) simply implies that $\sum_i \vec{F}_i = 0$ or sum of forces on all the particles in a closed system is zero. In particular, for a system of two particles, $\vec{F}_i + \vec{F}_2 = 0$: the force exerted by the first particle on the second is equal in magnitude and opposite in direction to that exerted by the second particle on the first. This is nothing but the Newton's third law.

Using Lagrange's equation, the condition (47a) results in

$$\sum_i \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}_i} = \frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} = 0 \tag{48}$$

Thus, we conclude that in a closed mechanical system, the vector

$$\vec{P} = \sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} = \sum_i \frac{\partial L}{\partial \vec{\upsilon}_i} = \sum_i m_i \vec{\upsilon}_i \tag{49}$$

which is total linear momentum of the system, remains constant during the motion. Since the motion of a single particle can be described by three cartesian coordinates, there would be three constants of motion, p_x, p_y and p_z, which are the three components of a linear momentum vector $\vec{p}$. In more general terms, we may make the following statement:

If the Lagrangian of a system (closed) is invariant w.r.t. a translation in a certain direction, then the linear momentum of the system in that direction is constant in time or constant of motion.

(iii) Conservation of Angular Momentum

Let us now derive the conservation law which follows from the isotropy of space which means that the mechanical properties of a closed system do not change when it is rotated as a whole in any manner in space. Let us, therefore consider an infinitesimal rotation of the system and obtain the condition for the Lagrangian to remain unchanged.

We shall use the vector $\delta\vec{\phi}\,(=\vec{n}\delta\phi)$ of infinitesimal rotation whose magnitude is the angle of rotation $\delta\phi$ and whose direction is that of the axis of rotation $\vec{n}$.

Let us find the resulting increment in the radius (or position) vector from an origin on the axis of rotation to any particle in the system undergoing rotation. The linear displacement of the end of the radius vector is related (see Fig. 2.3) to the angle of rotation by

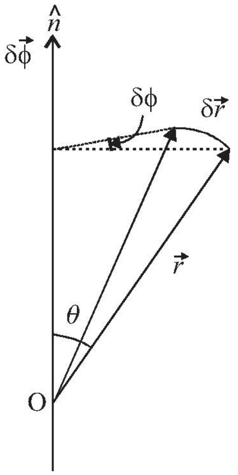

Fig. 2.3 Change of position vector under rotation of the system

$$|\delta\vec{r}| = r\sin\theta\,\delta\phi$$

The direction of $\delta\vec{r}$ is perpendicular to both $\vec{r}$ and $\vec{n}$. Therefore it can be written as

$$\delta\vec{r} = \delta\vec{\phi}\times\vec{r} = \delta\phi\vec{n}\times\vec{r} \tag{50}$$

When the system is rotated, not only the radius vectors but also the velocities of the particles change direction. Hence the velocity increment relative to a fixed coordinate system is

$$\delta \vec{\upsilon} = \delta \vec{\phi} \times \vec{\upsilon} \tag{51}$$

If the expressions (50) and (51) are substituted in the condition that the Lagrangian is unchanged by rotation, we get

$$\delta L = \sum_i \left(\frac{\partial L}{\partial \vec{r}_i} \cdot \delta \vec{r}_i + \frac{\partial L}{\partial \vec{\upsilon}_i} \cdot \delta \vec{\upsilon}_i \right) = \sum_i \left(\frac{\partial L}{\partial \vec{r}_i} \cdot \delta \vec{\phi} \times \vec{r}_i + \frac{\partial L}{\partial \vec{\upsilon}_i} \cdot \delta \vec{\phi} \times \vec{\upsilon}_i \right) = 0$$

Further replacing $\frac{\partial L}{\partial \vec{\upsilon}_i}$ by $\vec{p}_i$ and $\frac{\partial}{\partial}$ by $\dot{\vec{p}}_i$, we get

$$\delta L = \sum_i \left(\dot{\vec{p}}_i \cdot \delta \vec{\phi} \times \vec{r}_i + \vec{p}_i \cdot \delta \vec{\phi} \times \vec{\upsilon}_i \right) = 0$$

Now using the vector identity, $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A}$ and taking $\delta \vec{\phi}$ outside, we get

$$\delta \vec{\phi} \cdot \sum_i \left(\vec{r}_i \times \dot{\vec{p}}_i + \vec{\upsilon}_i \times \vec{p}_i \right) = \delta \phi \frac{d}{dt} \sum_i \vec{r}_i \times \vec{p}_i = \delta \phi \frac{d}{dt} \vec{n} \cdot \sum_i \vec{r}_i \times \vec{p}_i = 0$$

Since $\delta \phi$ is arbitrary, it follows that

$$\frac{d}{dt} \vec{n} \cdot \vec{M} = 0,$$

where
$$\vec{M} = \sum_i \vec{r}_i \times \vec{p}_i \tag{52}$$

We, therefore, conclude that $\vec{n} \cdot \vec{M}$ which is the component of angular momentum of the system along $\vec{n}$ is conserved in the motion of a closed system. Therefore, one infers that if $\delta \phi$ corresponds to a rotation of the system around some axis, then the angular momentum in that direction would be conserved.

It may be pointed out here that though the law of conservation of the angular momentum is valid for a closed system or in the absence of external field but it may hold good even for a system in an external field. The component of angular momentum along any axis about which the field is symmetrical is always conserved because of the fact that the mechanical properties of the system are unaltered by any rotation about that axis. The most important such case is that of the centrally or spherically symmetric field (central field), one in which the potential energy is function only of the distance from centre. In such a field, the component of angular momentum along any axis passing through the centre would be conserved. Another example is that of a homogeneous field in the z-direction. In such a field, the component M_z of the angular momentum is conserved.

We can summarize the above results by simply pointing out that if the system is invariant under translation along a given direction, the corresponding linear momentum is conserved. Similarly if the system is invariant under rotation about a given axis, the angular momentum in that direction is conserved. Thus the momentum conservation theorems are closely connected with the symmetry properties of the system (also see preceding paragraph). Also at the end it must be pointed out that every closed system has seven constants (or integrals) of motion: energy, three components of linear momentum and three components of angular momentum.

PROBLEMS

2.1 Prove that the shortest distance between two points in a plane is a straight line.

2.2 Let P$_1$ (x_1, y_1) and P_2 (x_2, y_2) be two fixed points in the xy-plane. Find the curve $x = x(y)$ passing through the points P_1 and P_2 which would generate by revolution about y-axis a surface of minimum area (see Fig. 2.4).

2.3 Show that the curve joining two points P_0 and P_1, along which a particle falling from rest under the influence of gravity travels, in the minimum time, is a cycloid with a cusp at the initial point p_0 at which the particle is released (see Fig. 2.5).

2.4 Show that the geodesics of a spherical surface are great circles (circles whose centers lie at the centre of sphere).

2.5 Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two fixed points in the xy-plane. Find the curve $y = y(x)$ passing through P_1 and P_2 which would generate by revolution about x-axis a surface of minimum area.

2.6 Show that the shortest distance between two points in space is a straight line.

2.7 Consider a bead of mass m sliding down from the top of a frictionless hoop of radius a. Find the radial force of the hoop on the bead using Lagrangian multiplier formalism.

SOLUTIONS

2.1 An element of arc length in the xy-plane is

$$ds = [(dx)^2 + (dy)^2]^{1/2} = [1 + (dy/dx)^2]^{1/2}\, dx$$

and therefore the total length of any curve going between points 1 and 2 is

$$I = \int_1^2 ds = \int_{x_1}^{x_2} \left(1 + y'^2\right)^{1/2} dx$$

The condition that the curve be the shortest path is that the integral I should be a minimum. This is an extremum problem as expressed by equ. (1) with

$$f = (1 + y'^2)^{1/2}$$

Substituting in the Euler - Lagrange's equation (9) with

$$\frac{\partial f}{\partial y} = 0, \ \frac{\partial f}{\partial y'} = ½(1 + y'^2)^{-½} \times 2y',$$

one gets

$$\frac{d}{dx}[y'/(1 + y'^2)^{½}] = 0$$

or $$y'/(1 + y'^2)^{½} = C$$

where C is a constant. Squaring both the sides and readjusting, one gets

$$y'^2/1 + y'^2 = C^2 \quad \text{or} \quad y'^2 = C^2/(1 - C^2)$$

or $$y' = a \tag{53}$$

where a is a constant related to C by

$$a = C/(1 - C^2)^{½}$$

Integrating equ. (53), we get

$$y = ax + b \tag{54}$$

where b is another constant of integration. Equ. (54) represents a straight line. Truly speaking, the straight line has only been proved to be an extremum path, but for this problem and from our common sense is also a minimum. The constants of integration, a and b can be determined by the condition that the curve passes through the two points, (x_1, y_1) and (x_2, y_2).

2.2 The area of the strip of the surface (see Fig. 2.4) is $da = 2\pi x\ ds = 2\pi\ x\ [(dx)^2 + (dy)^2]^{½}$ and hence the total area of the surface is

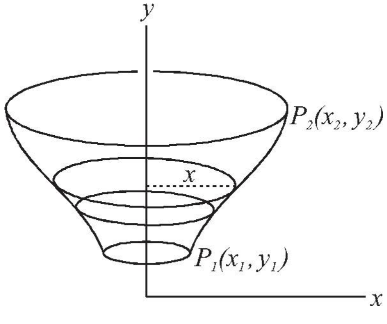

Fig. 2.4 Surface of revolution of minimum area. The points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are fixed

$$I = \int_1^2 da = 2\pi \int_{x_1}^{x_2} x\left(1 + y'^2\right)^{½} dx$$

To make the above integral a minimum (a maximum would not make any sense), the integrand

$$f = x(1 + y'^2)^{½}$$

must satisfy the Euler - Lagrange's equation (9). Here

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = x\ \tfrac{1}{2}(1 + y'^2)^{-1/2} \times 2y'$$

Putting above in equ. (9), one gets

$$d/dx(xy'/(1 + y'^2)^{1/2}] = 0$$

or $$xy'/(1 + y'^2)^{1/2} = a$$

where a is a constant of integration. Squaring the above equation and the rearranging, one gets

$$y'^2 = a^2/(x^2 - a^2)$$

or $$y' = a/(x^2 - a^2)^{1/2},$$

which on integration yields

$$y = a\int \frac{dx}{(x^2 - a^2)^{1/2}} + b = a \cosh^{-1} x/a + b$$

or $$x = a \cosh \frac{y - b}{a} \tag{55}$$

which is the equation of a catenary (from the Latin, catena means chain). This is truly speaking the shape of a chain under gravity, when it is held fix at two points. The constants a and b can be determined by the conditions that the curve passes through the two given end points. Note that in case we want to revolve the curve about x-axis, in that case the function f would become

$$f = y(1 + x'^2)$$

and the Euler - Lagrange's equation would read as

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

2.3 This brachistochrone (The word derives from the Greek brachistos (shortest) and chronos (time)) problem is famous in the history of mathematics because it was the analysis of this problem by John Bernoulli that led to the formal foundation of the calculus of variations.

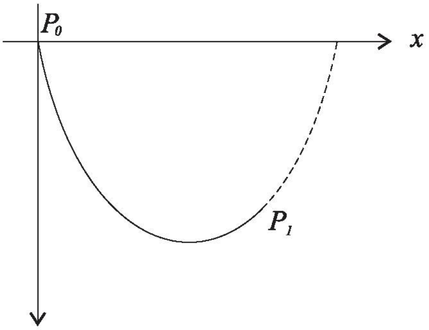

Fig. 2.5 Generation of cycloid y = $a(1 - \cos 2\theta)$

If v is the speed of particle along the curve from P_0 to P_1 (see Fig. 2.5), then the time required to fall an arc length ds is ds/v and hence the total time of descent from point P_o to point P_1 would be

$$\tau = \int_{P_0}^{P_1} ds/v$$

If y is measured down from the initial point of release, then from the conservation of energy theorem for the particle, one gets

$$½\, mv^2 = mg\, y$$

As a result,

$$\tau = \int_{P_0}^{P_1} [(dx)^2 + (dy)^2]^{½}/v = 1/(2g)^{½} \int_0^{y_1} (1 + x'^2)^{½}/y^{½} dy$$

and the function f in this case is identified as

$$f = [(1 + x'^2)/y]^{½}$$

as the factor $(2g)^{-1/2}$ does not affect the solution.

For τ to be extremum, the above function should satisfy the Euler-Lagrange's equation

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

which on substituting f becomes (as $\frac{\partial f}{\partial x} = 0$)

$$\frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

or $$\frac{\partial f}{\partial x'} = \text{const. } [1/(2a)^{1/2}]$$

or $$x'/[y(1 + x'^2)]^{1/2} = 1/(2a)^{1/2}$$

or $$x'^2 = y/(2a - y)$$

or $$x' = \frac{dx}{dy} = y^{1/2}/(2a - y)^{½}$$

or $$x = \int_0^y y^{½}/(2a - y)^{1/2} dy$$

To solve this, we put

$$y = 2a \sin^2 \theta \tag{56}$$

As a result,

$$x = 2a \int_0^\theta (1 - \cos 2\theta)\, d\theta$$

$$= 2a\left(\theta - \frac{\sin 2\theta}{2}\right) \tag{57}$$

Thus, we have

$$x = a\,(2\theta - \sin 2\theta) \tag{58}$$

$$y = a\,(1 - \cos 2\theta)$$

Equs. (58) are the parametric equations of a cycloid having x-axis as the basis and the concavity upwords. The constant a is the radius of the circle that generates the cycloid. Truly speaking, we have shown that the path of the cycloid insures a stationary value of τ, but it is obvious that extremum must be a minimum.

2.4 Note that geodesics are curves representing extremal paths between any two points, the paths being restricted to lie on a given surface, which in the present case is spherical.

The element of length ds, on the surface of a sphere of radius r, in the spherical polar coordinates, is given by

$$ds = [(dr)^2 + r^2\,(d\theta)^2 + (r \sin \theta\; d\phi)^2]^{1/2}$$

$$= r[(d\theta)^2 + \sin^2 \theta\;(d\phi)^2]^{1/2}$$

in which we have taken $dr = 0$.

The distance s from point 1 to point 2 is, therefore

$$s = \int_1^2 ds = r\int_{\theta_1}^{\theta_2} (1 + \sin^2\theta\phi'^2)^{1/2}\, d\theta,\ \phi' = d\phi/d\theta$$

Therefore, the function f, in this case would be given by

$$f = (1 + \sin^2 \theta\, \phi'^2)^{1/2} \tag{59}$$

If s is to be extremum, we must have

$$\frac{\partial f}{\partial \phi} - \frac{d}{d\theta}\frac{\partial f}{\partial \phi'} = 0 \tag{60}$$

since $\frac{\partial f}{\partial \phi} = 0$, the Euler - Lagrange's equ. (60) gives

$$\frac{d}{d\theta}\frac{\partial f}{\partial \phi'} = 0$$

or

$$\frac{\partial f}{\partial \phi'} = a \text{ (a constant)}$$

Therefore,

$\sin^2 \theta\, \phi'/(1 + \sin^2\theta\, \phi'^2)^{1/2} = a$

Squaring and rearranging the terms, one gets

$$\phi' = \frac{a}{\sin^2\theta(1-a^2\operatorname{cosec}^2\theta)^{1/2}} = \frac{a\operatorname{cosec}^2\theta}{(1-a^2-a^2\cot^2\theta)^{1/2}}$$

$$= \operatorname{cosec}^2\theta/(k^2-\cot^2\theta)^{1/2},\ k=(1-a^2)^{1/2}/a$$

The integration of the above equation yields

$$\phi = \sin^{-1}(\cot\theta/k)+\alpha \tag{61}$$

where k and α are constants

From equ. (61),

$$\cot\theta = k\sin(\phi-\alpha)$$

Multiplying both sides by a, one gets

$k\,(a\sin\theta\sin\phi\cos\alpha - a\sin\theta\cos\phi\sin\alpha) = a\cos\theta$

or $$bx+cy=z \tag{62}$$

where $b=-k\sin\alpha$ and $c=k\cos\alpha$.

The equ. (62) represents a plane passing through the origin (centre of the sphere) as no constant is occurring. Hence it would cut the surface of sphere in a great circle. One must note that the great circle route between two points, on a sphere is maximum as well as minimum depending on which way you go.

2.6 An element of arc length in space is

$$ds = [(dx)^2+(dy)^2+(dz)^2]^{1/2}$$

and therefore, the total length of the curve going between points 1 and 2 is

$$I = \int_1^2 ds = \int_1^2 (1+y'^2+z'^2)^{1/2}\,dx$$

The condition that the curve be shortest path is that the integral I should be minimum. This is a extremum problem as expressed by equ. (1) with

$$f = (1+y'^2+z'^2)^{1/2}$$

Substituting above in Euler - Lagrange's equations,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0,$$

and $$\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z'} = 0,$$

we get

$$\frac{d}{dx}\frac{y'}{(1+y'^2+z'^2)^{1/2}} = \frac{d}{dx}\frac{z'}{(1+y'^2+z'^2)^{1/2}} = 0$$

which means

$$y' = z' = \text{const.}$$

That is,

$$\frac{dy}{dx} = C_1 \text{ or } y = C_1x + C_3 \tag{63}$$

and
$$\frac{dz}{dx} = C_2 \text{ or } z = C_2x + C_4 \tag{64}$$

Adding (63) and (64), we get the required result.

2.7 Let θ be the angle from the vertical line through the centre of the hoop. The Lagrangian in this case would be given by

$$L = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mg\, r\cos\theta \tag{65}$$

where we have taken origin at the centre of the hoop.

The constraint of the hoop is given by

$$\psi(r, \theta) = r - a = 0 \tag{66}$$

The Lagrange's equations of motion in r and θ, that is,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda\frac{\partial \psi}{\partial r},$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial \psi}{\partial \theta},$$

are

$$-ma\dot{\theta}^2 + mg\cos\theta = \lambda, \tag{67}$$

$$ma^2\ddot{\theta} - mg\, a\sin\theta = 0 \tag{68}$$

where we have used $r = a$, $\ddot{r} = 0$, $\frac{\partial \psi}{\partial r} = 1$, and $\frac{\partial \psi}{\partial \theta} = 0$

From equ. (68),

$$\ddot{\theta} = g/a \sin\theta$$

Multiplying each side by $\dot{\theta}$ and integrating *w.r.t.* time gives

$$\dot{\theta}^2/2 = -g/a\cos\theta + C$$

The constant of integration C can be found from the condition that for $\theta = 0$, $\dot{\theta} = 0$, where upon

$$\dot{\theta}^2 = 2g/a(1 - \cos\theta)$$

Putting $\dot{\theta}$ from above into equ. (67), we get (for the constraint force),

$$\lambda = mg\cos\theta - 2\,mg(1 - \cos\theta) = mg(3\cos\theta - 2) \tag{69}$$

At the top ($\theta = 0$), this radial force is equal to the weight of the bead; at the angle whose cosine is 2/3, the force is zero and the bead falls of the hoop. It may be mentioned here that the result would be the same even if we consider a sphere of radius a instead of hoop.

3 CHAPTER

Integration of the Equation of Motion

3.1 MOTION IN ONE DIMENSION

The motion of a system having one degree of freedom is supposed to take place in one dimension. The Lagrangian of such a system in terms of cartesian coordinate x is given by

$$L = \tfrac{1}{2} m \dot{x}^2 - V(x) \tag{1}$$

For such a system, from the law of conservation of energy, the total energy is given by

$$H = E = \tfrac{1}{2} m \dot{x}^2 + V(x) \tag{2}$$

From above,

$$dx/dt = \sqrt{2\{E - V(x)\}/m}$$

which on integration yields,

$$t = (m/2)^{1/2} \int \frac{dx}{\{E - V(x)\}^{1/2}} + \text{constant} \tag{3}$$

Since the kinetic energy is essentially positive, the total energy always exceeds the potential energy or $E > V(x)$. That is, the motion can take place only in those regions of space in which $V(x) < E$. For example, let the function $V(x)$ be of the form shown in Fig. 3.1. If we draw in the figure a horizontal line corresponding to a given value of the total energy, we immediately find the possible regions of motion. In the given example, the motion takes place only in the range AB where $V(x) < E$ or in the range right to C where once again $V(x) < E$.

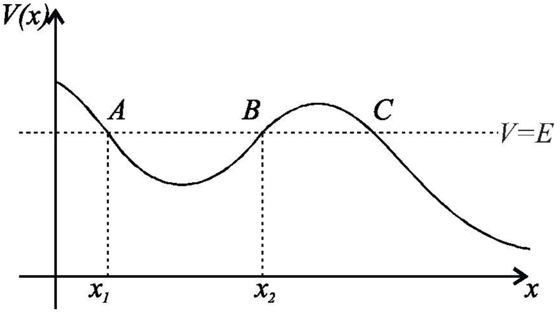

Fig. 3.1 $V(x)$ versus x

The points at which the potential energy equals the total energy, that is,

$$V(x) = E \tag{4}$$

give the limits of motion. These points are called turning points since the kinetic energy or effectively the velocity is zero there. If the region of motion is bounded by two such points, then the motion takes place in the finite region of space and is said to be finite. If the region of motion is limited only on one side or on neither, then the motion is infinite and the particle goes to infinity.

A finite motion in one dimension is oscillatory, the particle moving repeatedly back and forth between two points (in Fig. 3.1, in the potential well *AB* between the points x_1 and x_2). The period T of oscillation, that is, the time during which the particle passes from x_1 to x_2 and back, is twice the time from x_1 to x_2. From equ. (3)

$$T = (2\,m)^{1/2} \int_{x_1}^{x_2} dx/\sqrt{E - V(x)} \tag{5}$$

where x_1 and x_2 are roots of equ. (4) for the given value of energy E. No constant of integration would appear here as the integral is finite. Equ. (5) gives the period of oscillation as a function of the total energy of the particle.

Example 1. Find the period of oscillation of a particle of mass m suspended by a string of length l in a gravitational field, that is, of a simple pendulum as a function of the amplitude of oscillations.

The energy of the pendulum (see Fig. 1.4), is given by

$$E = \tfrac{1}{2}\,\mathrm{m}\, l^2 \dot{\theta}^2 - \mathrm{mg}\, l \cos\theta = -\mathrm{mg}\, l \cos\theta_0,$$

where θ is the angle between the string and the vertical and θ_0 the maximum value of θ, or the amplitude of oscillation.

The time period of the pendulum would be the time required to go from $\theta = 0$ to $\theta = \theta_0$ multiplied by four. Therefore, on substituting the appropriate values in equ. (3), one gets

$$\begin{aligned}
T &= 4\,(m/2)^{1/2} \int_0^{\theta_0} l d\theta/\sqrt{-mgl\cos\theta_0 - (-mgl\cos\theta)} \\
&= 4\,(l/2\mathrm{g})^{1/2} \int_0^{\theta_0} d\theta/\sqrt{\cos\theta - \cos\theta_0} \\
&= 4\,(l/2\mathrm{g})^{1/2} \int_0^{\theta_0} d\theta/\sqrt{(1 - 2\sin^2\theta/2) - (1 - 2\sin^2\theta_0/2)} \\
&= 2\,(l/\mathrm{g})^{1/2} \int_0^{\theta_0} d\theta/\sqrt{\sin^2\theta_0/2 - \sin^2\theta/2} \\
&= 2\,(l/\mathrm{g})^{1/2} \int_0^{\theta_0} \frac{d\theta/\sin\theta_0/2}{\sqrt{1 - \sin^2\theta/2/\sin^2\theta_0/2}}
\end{aligned}$$

Substituting $\sin \xi = \sin \theta/2 \,/\, \sin \theta_0/2$ or $\cos \xi \, d\xi = ½ \cos \theta/2 \, d\theta/\sin \theta_0/2$ = ½ $(1 - \sin^2 \xi \sin^2 \theta_0/2)\, d\theta/\sin^2 \theta_0/2$ into equ. (6), one gets

$$T = 2(l/g)^{1/2} \int_0^{\pi/2} \frac{2\cos\xi d\xi / \sqrt{1-\sin^2\xi \sin^2\theta_0/2}}{\sqrt{1-\sin^2\xi}}$$

$$= 4(l/g)^{1/2} \int_0^{\pi/2} d\xi / \sqrt{1-\sin^2\xi \sin^2\theta_0/2}$$

$$= 4(l/g)^{1/2} K(\sin\theta_0/2)$$

where

$$K(\sin\theta_0/2) = K(k) = \int_0^{\pi/2} d\xi / \sqrt{1-k^2\sin^2\xi}$$

is a complete elliptical integral of the first kind.

For oscillations to be small

$k = \sin \theta_0/2 \approx \theta_0/2 << 1$, the expansion of $K(k)$ gives

$$K(k) = \pi/2\, [(1 + (½)^2 k^2 + (3/8)^2 k^4 + ...]$$

Substituting above into equ. (7), one gets

$$T = 2\pi(l/g)^{½}\,(1 + \theta_0^2/16 + ...) \quad (8)$$

The first term corresponds to the familiar result for the time period of a simple pendulum.

3.2 THE REDUCED MASS

Let us now consider the motion of two interacting particles of masses m_1 and m_2 (see Fig. 3.2), that is, a two body problem (like sun-earth or nucleus - electron). Assume that the potential energy of interaction of two particles is a function of the distance between them or that means depends on the magnitude of the difference in their radius or position vectors, $|\vec{r}_1 - \vec{r}_2|$. Such a system has six degrees of freedom and hence six independent generalized coordinates. We chose these to be the three components of the radius vector $\vec{R}$ to the centre of mass (com), plus the three components of the difference or relative position vector.

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad (9a)$$

Further, if we assume origin to be at the com (see Fig. 3.3) that is,

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0, \quad (9b)$$

we get

$$m_1\vec{r}_1 + m_2(\vec{r} + \vec{r}_1) = 0$$

or

$$\vec{r}_1 = -m_2\vec{r}/(m_1 + m_2) \quad (10a)$$

and similarly

$$\vec{r}_2 = m_1 \vec{r}/(m_1 + m_2) \tag{10b}$$

where $\vec{r}_1$ and $\vec{r}_2$ are the radii vectors of the two particles relative to the com.

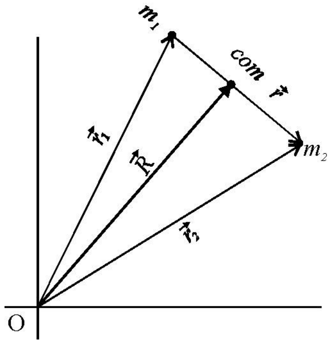

Fig. 3.2 Coordinates for the two body problem

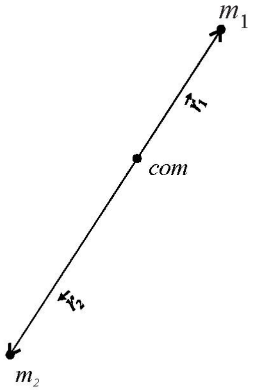

Fig. 3.3 Position of two particles with origin at Com

The Lagrangian of such a system is given by

$$L = \tfrac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - V\left(\left|\vec{r}_1 - \vec{r}_2\right|\right) \tag{11}$$

Substitution of (10) into above equation gives

$$\mathrm{L} = [\tfrac{1}{2}m_1 m_2^2/(m_1+m_2)^2 + \tfrac{1}{2}m_2 m_1^2/(m_1+m_2)^2]\,\dot{\vec{r}}^{\,2} - V(r)$$

$$= \tfrac{1}{2}\mu \dot{\vec{r}}^{\,2} - V(r) \tag{12}$$

where $\mu = m_1 m_2/(m_1 + m_2)$, (13)

is called the reduced mass. Note that the Lagrangian (12) is exactly what would be expected if we had a fixed centre of force with a single particle of mass μ moving at a distance r in an external field $V(r)$. Thus the problem of two interacting particles is equivalent to the motion of a single particle of mass μ moving in an external field such that its potential energy depends only on the distance r from some fixed point. This is called a central field. The force acting on the particle is $\vec{F} = \vec{F}(\vec{r}) = -\partial V(r)/\partial \vec{r} = -(dV/dr)\,\vec{r}/r = f(r)\,\vec{r}/r$; its magnitude is function of r only and its direction is every where that of the radius vector. If in some region $dV/dr > 0$ or $f(r) < 0$, the force points to the centre of the field, being a force of attraction in that region. In the opposite case, the force is repulsive.

3.3 MOTION IN A CENTRAL FIELD

Let us restrict ourselves to conservative central forces where the potential energy V is a function of radial distance r only, so that the force is always along $\vec{r}$. By the result of the preceding section, we need consider only the problem of a single particle of mass μ moving about a fixed centre of force, which would be taken as the origin of the coordinate system. Since the potential energy involves only the radial distance, the problem has spherical symmetry, that is, any rotation about any fixed axis can have no effect on the solution.

The torque exerted by the central force on the particle is

$$\vec{N} = \vec{r} \times \vec{F}(\vec{r}) = \vec{r} \times f(r)\hat{r} = rf(r)\hat{r} \times \hat{r} = 0 = d\vec{M}/dt$$

which amounts to the fact that the angular momentum $\vec{M}$ of the particle about any axis through the centre of force is conserved. The angular momentum of a single particle (as the problem has already been reduced to the motion of a single particle) is $\vec{M} = \vec{r} \times \vec{p}$. Since $\vec{M}$ is perpendicular to $\vec{r}$, the constancy of $\vec{M}$ infers that throughout the motion, the radius vector of the particle lies in a plane perpendicular to $\vec{M}$. Thus the path of a particle in a central field lies in a plane whose normal is parallel to $\vec{M}$ or in other words, the central force motion is always motion in a plane. Using polar coordinates in that plane, we can write the Lagrangian as

$$L = T - V = \tfrac{1}{2}\mu\left(\dot{x}^2 + \dot{y}^2\right) - V(r) = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \tag{14}$$

as $x = r\cos\theta, y = r\sin\theta$ or $\dot{x} = -r\sin\theta\dot{\theta} - \dot{r}\cos\theta,\ \dot{y} = r\cos\theta\dot{\theta} + \dot{r}\sin\theta$

As the Lagrangian does not contain the coordinate θ, it is cyclic. For such a coordinate, we have from Lagrange's equation $d/dt\ \partial L/\partial\dot{\theta} = \partial L/\partial\theta = 0$ or the corresponding generalized momentum

$$p_\theta = \partial L/\partial\dot{\theta} = \mu\, r^2\dot{\theta} = l \tag{15}$$

which is the angular momentum, is conserved or is an integral of motion.

From the above equation, it also follows that

$$d/dt\,(\tfrac{1}{2}\, r^2\dot{\theta}) = 0 \tag{16}$$

The factor ½ is introduced because of the fact that $\tfrac{1}{2}\, r^2\dot{\theta}$ is the area swept out by the radius vector per unit time. This interpretation follows from the Fig. 3.4 because dA is the area swept out by the radius vector in time dt, that is,

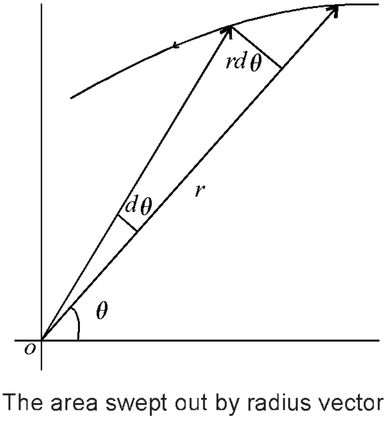

Fig. 3.4 The area swept out by radius vector in time dt.

$$dA = \tfrac{1}{2}\, r(r\, d\theta)$$

and hence

$$dA/dt \text{ (areal velocity)} = \tfrac{1}{2}\, r^2\dot{\theta}$$

The conservation of angular momentum is thus equivalent to the statement that the areal velocity is constant. That means in equal times the radius vector of the particle sweeps out equal areas. This is known as Kepler's second law of planetary motion.

The complete solution of the problem of the motion of a particle in a central field is most simply obtained by starting from the law of conservation of energy and angular momentum, without writing the equations of motion themselves. Expressing $\dot{\theta}$ in terms of l from equ. (15) and substituting in the expression for the total energy (another constant of motion as the force is conservative. Indeed the Lagrangian (14) does not depend explicitly on time), we get

$$\begin{aligned} E &= \tfrac{1}{2}\,\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \\ &= \tfrac{1}{2}\,\mu\,\dot{r}^2 + l^2/2\mu r^2 + V(r) \qquad (17) \end{aligned}$$

Hence

$$\dot{r} = dr/dt = 2/\mu \ \ (E - V - l^2/2\mu r^2) \qquad (18)$$

or on integration,

$$t = \int dr/\sqrt{2/\mu(E - V - l^2/2\mu r^2)} + \text{constant} \qquad (19)$$

The above equation gives t as a function of r, which can be inverted to give, the distance r from the centre as a function of t.

Now from equ. (15),

$$d\theta = l\, dt/\mu r^2$$

After substituting the value of dt from (18) and integrating, we get

$$\theta = \int (l dr/r^2)/\sqrt{2/\mu(E - V - l^2/2\mu r^2)} + \text{constant} \qquad (20)$$

This gives relation between r and θ, that is, the equation of path. The equs. (19) and (20) give general solution of the problem.

The equ. (17) shows that the radial part of the motion can be regarded as taking place in one-dimension in a field where the effective (or fictitious) potential energy is

$$V_{\text{eff}} = V' = V(r) + l^2/2\mu r^2 \qquad (21)$$

The quantity $l^2/2\mu r^2$ is called the centrifugal potential energy as $-d/dr$ of this gives the centrifugal force.

3.4 THE EQUIVALENT ONE - DIMENSIONAL PROBLEM, AND CLASSIFICATION OF ORBITS

(a) Case of an attractive inverse - square law of force

Though we have solved the one-dimensional problem formally but practically speaking the integrals (19) and (20) are usually unmanageable and in any specific case it is usually more convenient to perform the integration in some other fashion.

As an example, consider a specific case of an attractive inverse - square law of force:

$$f = -k/r^2$$

where k is positive and the minus sign ensures that the force is toward the centre of force (Gravitational attraction and Coulombian interactions are well known examples of this kind of force).

The potential energy of this force is,

$$V(r) = -k/r,$$

and the corresponding effective (or fictitious) potential energy is

$$V' = -k/r + l^2/2\mu r^2,$$

which is displayed in Fig. 3.5 as a function of r (solid line). The two dotted lines represent the separate components $-k/r$ and $l^2/2\mu r^2$.

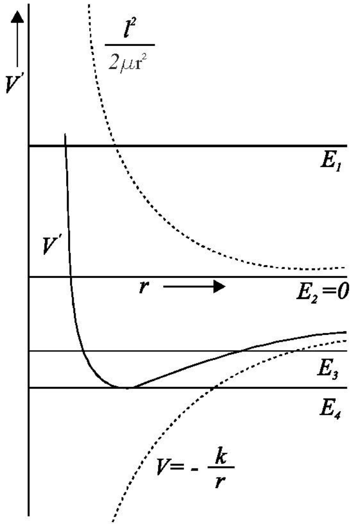

Fig. 3.5 The equivalent one-dimensional potential for an attractive inverse-square law of force.

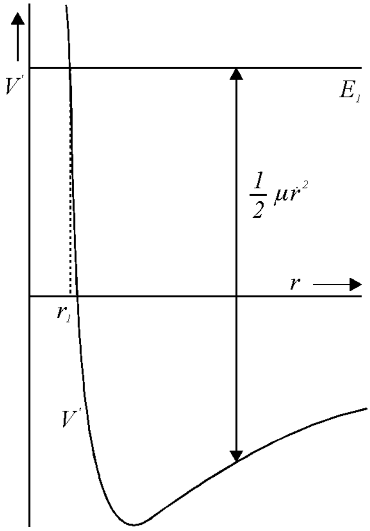

Fig. 3.6 Unbounded motion at positive energies for inverse-square law of force

Consider now the motion of a particle having energy $E = E_1$ as shown in Fig. 3.6. Note that for $r < r_1$, $V' > E_1$, the kinetic energy becomes negative, corresponding to an imaginary velocity. This amounts to the fact that the particle can never come closer than r_1 (see Fig. 3.6). On the other hand, there is no upper limit to the possible value of $r(E_1 > V'$, for all $r > r_1)$, hence the orbit is not bounded. The distance between E_1 and V' is $\frac{1}{2}\mu\dot{r}^2$, that is, proportional to the radial velocity and becomes zero at $r = r_1$ $(E_1 = V')$, called the turning point. The angular part of the kinetic energy is $l^2/2\mu r^2$ $(=\frac{1}{2}\mu\, r^2\dot{\theta}^2)$, which is just the difference $V' - V(r)$; this quantity rises to maximum at the turning point r_1.

To summarize, one can say that a particle with energy E_1 will come in from infinity, strike the repulsive centrifugal barrier and repelled and there after travel back out to infinity (See Fig. 3.7).

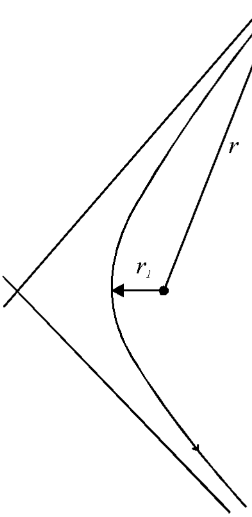

Fig. 3.7 The orbit for E=E_1 corresponding to unbounded motion.

For a particle having energy $E = E_2 = 0$ (see Fig. 3.5), a roughly similar behaviour to that of at $E = E_1$ is observed. But for a particle having negative energy $E = E_3$ (see Fig. 3.8), in addition to a lower bound r_1, there is also a maximum value r_2 where $E_3 = V'$. The motion is, therefore said to be bounded and there are two turning points r_1 and r_2, also known as apsidal distances. This does not necessarily mean that the orbits are closed. All that can be said is that they are bounded, contained between two circles of radius r_1 and r_2 with turning points always lying on the circles (see Fig. 3.9).

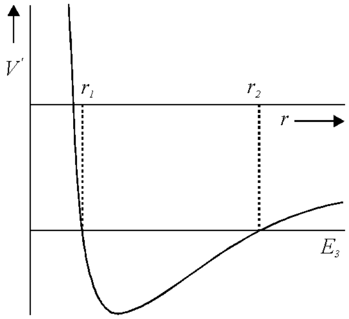

Fig. 3.8 The equivalent one-dimensional potential for inverse-square law of force illustrating bounded motion at negative energies.

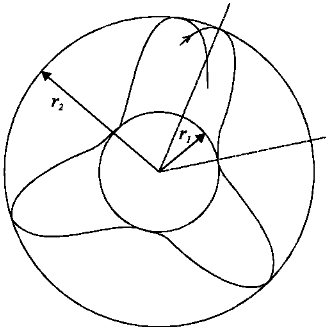

Fig. 3.9 The nature of orbits for bounded motion.

For a particle having energy $E = E_4$ at the minimum of V' (see Fig. 3.10), the two bounds coincide. In such a case, the motion is possible only at one radius (from equ. 17, at $E_4 = V'$, $\dot{r} = 0$) $r = r_1$ and the orbit is a circle.

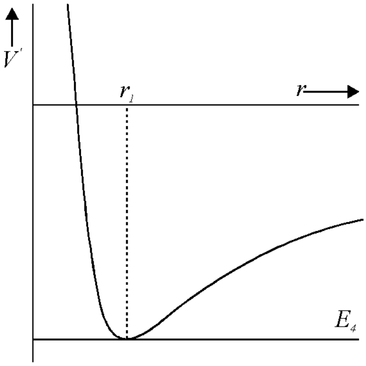

Fig. 3.10 The equivalent one-dimensional potential for inverse-square law of force illustrating the condition for circular orbit.

It may be noted that the case $E < E_4$ does not correspond to a physically possible motion because then the kinetic energy (or effectively $\dot{r}^2$) would have to be negative or $\dot{r}$ is imaginary.

It is necessary to mention here that all of the above discussion of the orbits for various energies has been at one value of the angular momentum. Changing l changes the quantitative details of the V' curve but will not effect the general classification of the types of orbits.

For the attractive inverse - square law of force discussed above, we shall see (Sec. 3.6) that the orbit for E_1 is hyperbola, for E_2 a parabola, for E_3 an ellipse and for E_4 a circle.

(b) Case of attractive force $f = -3a/r^4$ (or attractive potential $V(r) = -a/r^3$)

The energy diagram in this case is shown in Fig. 3.11. For an energy $E = E_1$, there are two possible types of motions depending upon the initial value of r (say equal to r_0). If r_0 is less than r_1, the motion will be bounded and r will always remain less than r_1. If r is initially greater than r_2, then it will remain so and the motion is unbounded.

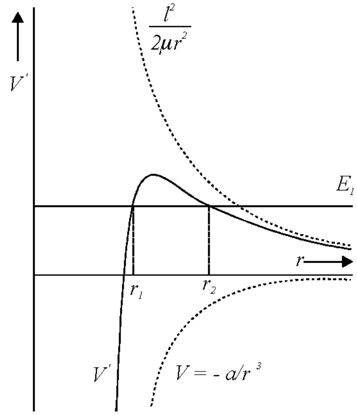

Fig. 3.11 Equivalent one-dimensional potential for an attractive $1/r^4$ law of force.

(c) Case of linear restoring force (isotropic harmonic oscillator)

$$f = -kr,\ V = \tfrac{1}{2} kr^2$$

For zero angular momentum corresponding to motion along a straight line, $V' = V$ and the situation is shown in Fig. 3.12. For any positive energy the motion is bounded and as we know is simple harmonic. For $l \neq 0$, we have the state of affairs as shown in Fig. 3.13. The motion is always bounded for all physically possible energies and does not pass through the centre of force. The x and y components of the force in this case are

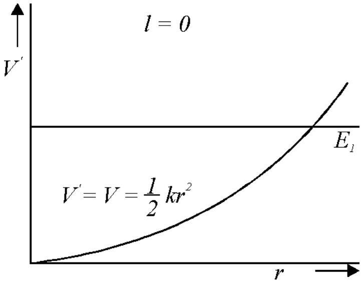

Fig. 3.12 Effective potential energy for zero angular momentum.

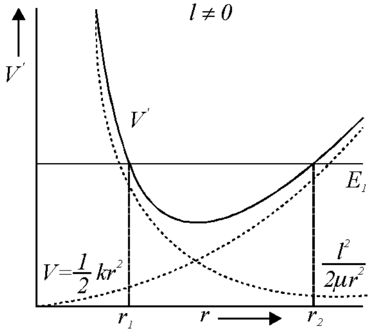

Fig. 3.13 The equivalent one-dimensional potential for a linear restoring force.

$$f_x = \mu \ddot{x} = -kx, f_y = \mu \ddot{y} = -ky$$

The total motion is the resultant of two simple harmonic oscillations of same frequency $(k/\mu)^{1/2}$ at right angle, and hence leads to an elliptic orbit.†

3.5 THE VIRIAL THEOREM

It is concerned with the time averages of various mechanical quantities. Consider a system of particles with position vectors $\vec{r}_i$. Let $\vec{F}_i$ be the total force acting on the ith particle. The fundamental equations of motion are then

$$\vec{F}_i = \dot{\vec{p}}_i \tag{22}$$

---

† From above equations,

$\ddot{x} + \omega^2 x = 0, \quad \ddot{y} + \omega^2 y = 0, \quad \omega^2 = k/\mu$

The solutions of these equations are

$x = a \sin \omega t, \quad y = b \cos \omega t$

where a and b are amplitudes of the component oscillators having the same frequency but differing in phase by $\pi/2$. Squaring both sides of the above equations and adding, we get

$x^2/a^2 + y^2/b^2 = 1$

which is equation of an ellipse with its major and minor axes coinciding with the direction of the two oscillations (in x and y directions) and shose semi major and minor axes are a and b, respectively.

Let us consider a quantity

$$G = \sum_i \vec{p}_i \cdot \vec{r}_i$$

where the summation is over all the particles in the system. The total time derivative of this quantity is given by

$$dG/dt = \sum_i \vec{p}_i \cdot \dot{\vec{r}}_i + \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i$$

$$= \sum_i m_i \vec{v}_i \cdot \vec{v}_i + \sum_i \vec{F}_i \cdot \vec{r}_i$$

$$= 2T + \sum_i \vec{F}_i \cdot \vec{r}_i \qquad (23)$$

where T denotes the kinetic energy of the system

The time average of above equation over a time interval τ is obtained by integrating both sides with respect to t from 0 to τ, and then dividing by τ:

$$1/\tau \int_0^\tau dG/dt \;\, dt = \overline{\frac{dG}{dt}} = 2\overline{T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i}$$

or $$2\overline{T} + \sum_i \vec{F}_i \cdot \vec{r}_i = 1/\tau\, [G(\tau) - G(0)] \qquad (24)$$

If the motion is periodic, that is, all coordinates repeat after a certain time τ, called the time period of motion, then the r.h.s. of the above equation vanishes. Note that if the motion is not periodic but bounded, by choosing τ sufficiently large, the r.h.s. of equ. (24) can be made as small as desired (or can be made to vanish). Therefore, in both the cases, we get

$$2\overline{T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i} = 0$$

or $$\overline{T} = -\tfrac{1}{2}\overline{\sum_i \vec{F}_i \cdot \vec{r}_i} \qquad (25)$$

The above equation is known as the virial theorem and the r.h.s. is called the virial of the system by Clausius.

If the forces are conservative, that is, derivable from a potential function, then the theorem or equ. (25) becomes

$$\overline{T} = \tfrac{1}{2}\, \overline{\sum_i \vec{\nabla} V \cdot \vec{r}_i} \qquad (26)$$

and for a single particle moving under a central force, it reduces to

$$\overline{T} = \tfrac{1}{2}\, \overline{\partial V/\partial r \; \vec{r}\,/\,r \cdot \vec{r}} \; = \tfrac{1}{2}\overline{\partial V/\partial r\, r} \qquad (27)$$

If V is a power law function of r, that is

$$V = a\, r^{n+1}$$

where exponent is chosen such that the force law goes as r^n, then

$$\frac{\partial V}{\partial r} r = a\,(n+1) r^n\, r = (n+1)V$$

and hence the equ. (27) becomes

$$\bar{T} = \frac{n+1}{2}\bar{V} \tag{28}$$

Note that the above equation can directly be obtained from Euler's theorem if V is a homogenous function in r of degree $n + 1$,

Further for the special case of inverse-square law of force n being-2 the Virial theorem takes on a well known form

$$\bar{T} = -\bar{V}/2 \tag{29}$$

Further more, for the case of small oscillations (V is a homogenous function of degree 2 in r), $n = 1$ and hence from equ. (28),

$$\bar{T} = \bar{V} \tag{30}$$

that is, the mean values of the kinetic and potential energies are equal.

3.6 THE DIFFERENTIAL EQUATION FOR THE ORBIT

We would consider here the inverse-square law (or effectively the Kepler problem) which is the most important of all the central force laws, and hence deserves a detailed treatment. For the said case, the force and potential are given by

$$f = -k/r^2, \qquad V = -k/r \tag{31}$$

From equ. (20), that is from the relation between r and θ (or equation of path), we have

$$\theta = \int \frac{dr/r^2}{(2\mu E/l^2 - 2\mu V/l^2 - 1/r^2)^{1/2}} + \text{constant}$$

Changing the variable of integration to $u = 1/r$ or $du = -dr/r^2$, we get

$$\theta = \theta' - \int du/(2\,\mu E/l^2 - 2\,\mu\, V/l^2 - u^2)^{1/2}$$

where θ' is a constant of integration which is to be determined from the initial conditions. Substituting (31) in the above differential equation for the orbit, we get

$$\theta = \theta' - \int du/(2\,\mu E/l^2 + 2\,\mu k u/l^2 - u^2)^{1/2} \tag{32}$$

The indefinite integral appearing in the above equation, is of the standard form,

$$\int dx/(a_1 + b_1 x + dx^2)^{1/2} = 1/(-d)^{1/2} \cos^{-1}\left(-\frac{b_1 + 2d\,x}{q^{1/2}}\right) \tag{33}$$

where

$$q = b_1^2 - 4\,a_1 d$$

Comparing (32) and (33), one gets

$$a_1 = 2\,\mu\, E/l^2,$$

$$b_1 = 2\,\mu\, k/l^2,$$

$$d = -1 \tag{34}$$

and hence the discriminant q is,

$$q = (2\,\mu k/l^2)^2 + 4 \times 2\,\mu E/l^2$$

$$= (2\,\mu k/l^2)^2\,(1 + 2El^2/\mu\,\mathrm{k}^2) \tag{35}$$

with these substitutions, equ. (32) becomes

$$\theta = \theta' - \cos^{-1}\left[-\frac{2\mu k/l^2 + 2(-1)u}{\{(2\mu k/l^2)^2\,(1+2El^2/\mu k^2)\}^{1/2}}\right]$$

Taking 2 $\mu k/l^2$ common from the numerator and as well as from the denominator and then on cancellation, we get

$$\theta = \theta' - \cos^{-1}\,(l^2\,u/\mu k - 1)/(1 + 2\,E\,l^2/\mu k^2)^{1/2}$$

Finally putting $u = 1/r$ in the above equation, the equation of orbit is found to be

$$1/r = \mu k/l^2\,\{1 + (1 + 2El^2/\mu k^2)^{1/2}\cos\,(\theta - \theta') \tag{36}$$

Now the general equation of a conic with one focus at the origin is

$$1/r = C\,\{1 + \in \cos\,(\theta - \theta')\} \tag{37}$$

where, $\in$ is the eccentricity of the conic section and 2C is called the latus rectum of the orbit. By comparison with equ. (36), it follows that the orbit is always a conic section with the eccentricity

$$\in = (1 + 2El^2/\mu k^2)^{1/2},\ \text{and}\ C = \mu k/l^2 \tag{38}$$

The nature of the orbit depends upon the magnitude of $\in$ and if

$\in > 1 \Rightarrow E > 0$, hyperbola

$\in = 1 \Rightarrow E = 0$, parabola

$\in < 1 \Rightarrow E < 0$, ellipse

$\in = 0 \Rightarrow E = -\mu k^2/2l^2$, circle

This classification agrees with the qualitative discussion of the orbits on the basis of the effective potential energy V' diagram in the case of equivalent one-dimensional problem. The condition for the circular motion appears in some what different form but it can be shown that $E = -\mu k^2/2l^2$ corresponds to a circular orbit.

For circular motion T and V are constant in time, and therefore, from the virial theorem (see equ. 29).

$$T = -V/2$$

Now

$$E = T + V = V/2 = -k/2r$$

Putting $E = -\mu k^2/2l^2$ in the above equation, one gets

$$-\mu k^2/2l^2 = -k/2r$$

Further putting $l = \mu r^2\dot{\theta}$, one gets

$$k/r^2 = \mu r\dot{\theta}^2 = \mu v^2/r,$$

the condition for the circular orbit

To attach physical meaning to θ' in equ. (37), let us differentiate r w.r.t. θ:

$$dr/d\theta = \in/C \sin(\theta - \theta')/\{1 + \in \cos(\theta - \theta')\}^2$$

which is equal to zero when $\theta = \theta'$, and the vanishing of $dr/d\theta$ marks the turning points on the orbit, called apsidal points where r reaches its maximum or minimum values. Orienting x-axis to pass through such a turning point, we can choose $\theta' = 0$, the angle $\theta = \pi$ would then give another turning point.

3.7 PERIOD OF MOTION IN ELLIPTICAL ORBIT

From the conservation of angular momentum, the areal velocity is constant and is given by

$$dA/dt = \tfrac{1}{2}\, r^2\dot{\theta} = l/\, 2\mu, \tag{39}$$

where we have put the value of $\dot{\theta}$ from $\mu r^2\, \dot{\theta} = l$.

The area of the orbit A, can be found by integrating (39) over a complete period τ,

$$A = \int_0^{\tau} dA/dt\, dt = \int_0^{\tau} l/2\mu \;\; dt = l\tau/2\mu$$

Hence the period τ is

$$\tau = 2\, \mu A/l \tag{40}$$

Area of the ellipse is

$$A = \pi\, ab,$$

where a and b are the semi major and semi minor axes, respectively.

From Fig. 3.14, for an elliptical orbit,

$$2a = r_{\text{min}} + r_{\text{max}}$$

where r_{min} and r_{max} are two absidal distances (or r_1 and r_2 shown in Fig. 3 or are roots of equation, $E = V'$ or $E - l^2/2\mu r^2 + k/r = 0$ or $r^2 + k/E\;\, r - l^2/2\mu E = 0$).

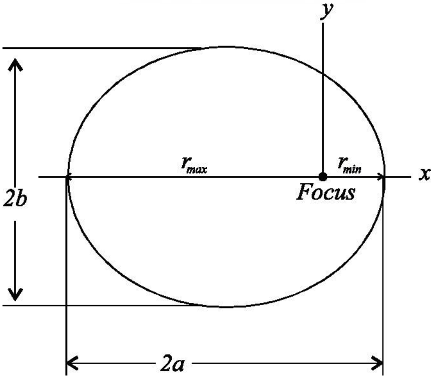

Fig. 3.14 Elliptical orbit corresponding to E < 0 (or $\in$<1).

After putting cos $(\theta - \theta') = +1$ and -1, respectively in equ. (37), one gets

$$2a = \frac{1}{C(1+\in)} + \frac{1}{C(1-\in)} = \frac{2}{C(1-\in^2)}$$

$$= \frac{2}{\mu k/l^2\ (1 - 1 - 2El^2/\mu k^2)} = -k/E$$

or $$a = -k/2E \tag{41}$$

By definition of eccentricity, the semi minor and semi major axes are related by the formula,

$$b = a(1 - \in^2)^{1/2}$$

$$= a(1 - 1 - 2El^2/\mu k^2)^{1/2}$$

$$= a(-2l^2/\mu k^2 \times -k/2a)^{1/2}$$

where we have put the value of E from equ. (41)

Therefore,

$$b = a^{1/2}(l^2/\mu k)^{1/2}$$

and the period, therefore is

$$\tau = 2\,\mu A/l = 2\,\mu\,\pi\,ab/l$$

$$= 2\pi\, a^{3/2}\,(\mu/k)^{1/2} \tag{42}$$

or $$\tau^2\,\alpha\,a^3 \tag{43}$$

This states that, other things being equal, the square of the period is proportional to the cube of the semi major axis, and this conclusion is often referred to as the third law of Kepler††.

Truly speaking Kepler was concerned with the specific problem of planetary motion in the gravitational field of the sun. In this case, the gravitational force of attraction between sun and planet is

$$f = -Gm_p m_s/r^2$$

where m_p and m_s are the masses of planet and sun, respectively. Hence, the constant k is

$$k = Gm_p m_s \tag{44}$$

Under these conditions, equ. (42) becomes

$$\tau = 2\pi\, a^{3/2}\,[\{m_p m_s/(m_p + m_s)\}/Gm_p m_s]^{1/2}$$

$$= 2\pi\, a^{3/2}/\sqrt{G(m_p + m_s)} \tag{45a}$$

If we neglect the mass of planet compared to sun,

[††] The first law of Kepler, that the planets move in elliptical orbits about the sun at one focus, and the third law are restricted to the inverse square law of force. Whereas the second law, is true for any central force motion.

$$\tau = 2\,\pi\, a^{3/2}/\sqrt{Gm_s} \tag{45b}$$

Note that the relation (45a) is true in general, whereas the relation (45b) is true only in the cases in which the mass of the planet is negligible compared to the sun.

3.8 EQUATION OF ORBIT-EQUATION OF MOTION METHOD

The Lagrangian in the central force field is given by

$$L = T - V = \tfrac{1}{2}\,\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

$$= \tfrac{1}{2}\,\mu\dot{r}^2 + l^2/2\mu r^2 - V(r)$$

where we have substituted the value of $\dot{\theta}^2$ from equ. (15). As a result, the Lagrange's equation of motion,

$$\partial L/\partial r - d/dt\; \partial L/\partial \dot{r} = 0,$$

takes the form,

$$\mu\ddot{r} - l^2/\mu r^3 = f(r) \tag{46}$$

We truly speaking want a relation between r and θ (or equation of the orbit). To do so, let us first find d^2/dt^2.

Now

$$d/dt = d/d\theta\; d\theta/dt = l/\mu r^2\; d/d\theta$$

and hence

$$d^2/dt^2 = d/d\theta\; (l/\mu r^2\; d/d\theta)\; d\theta/dt$$

$$= l/\mu r^2\; d/d\theta\; (l/ur^2\;\; d/d\theta)$$

where we have substituted the value of $\dot{\theta}$ from equ. (15).

The above relation is used to convert the equation of motion (46) into a differential equation for the orbit. Doing so, one gets

$l/r^2\; d/d\theta\; (l/\mu r^2\; dr/d\theta) - l^2/\mu r^3 = f(r)$

Noting that

$$1/r^2\; dr/d\theta = -d(1/r)/d\theta,$$

and then changing the variable to $x = 1/r$, the above equation reduces to

$$d^2u/d\theta^2 + u = -\mu/l^2\; u^2 f(1/u), \tag{47}$$

called Binet's equation

If the force field obeys the inverse square law (corresponding to Kepler problem),

$$f = -k/r^2 = -ku^2 \tag{48}$$

Substituting (48) into (47), one gets

$$d^2\, u/d\theta^2 + u = \mu k/l^2 \tag{49}$$

Introducing a variable,

$$y = u - \mu k/l^2, \qquad (49')$$

the above equation reduces to

$$d^2y/d\theta^2 + y = 0,$$

which is a second order differential equation and has a solution,

$$y = A\cos(\theta - \theta'), \qquad (50)$$

A and θ' being constants.

Substituting value of y back in (49′) and then putting $u = 1/r$, we get

$$1/r = \mu k/l^2 \{1 + \in \cos(\theta - \theta'), \qquad (51)$$

where

$$\in = Al^2/\mu k$$

Note that the above equation (of a conic section with origin at the focus) is similar to equ. (37), but the advantage of obtaining the said equation is that if the equation of the orbit is known, the force law can be obtained easily.

[Since $\mu = m_1 m_2/(m_1 + m_2)$ and in most of the cases $m_1 >> m_2, \mu = m_2$ ($= m$, the mass of the particle). Therefore, we use m instead of μ unless condition in the problem dictates otherwise]

Example 2. Find the central force field in which a particle describes $r = a(1 + \cos\theta)$ orbit . The example would illustrate the use of Binet's equation for finding the force law if the path of particle is known.

In the given case,

$$u = 1/r = \frac{1}{a(1+\cos\theta)}.$$

As a result,

$$du/d\theta = \frac{a(1+\cos\theta)\times 0 - 1\times(-a\sin\theta)}{a^2(1+\cos\theta)^2}$$

$$= \frac{\sin\theta}{a(1+\cos\theta)^2} = \frac{2\cos\theta/2\,\sin\theta/2}{a\times 4\cos^4\theta/2} = \frac{\sin\theta/2}{2a\cos^3\theta/2}$$

and hence

$$d^2u/d\theta^2 = \frac{2a\cos^3\theta/2\;1/2\,\cos\theta/2 - \sin\theta/2\times 2a\times 3\cos^2\theta/2(-\sin\theta/2)\times 1/2}{4a^2\cos^6\theta/2}$$

$$= \frac{(\cos^4\theta/2 + 3\sin^2\theta/2\,\cos^2\theta/2)}{4a\cos^6\theta/2} = \frac{(1+2\sin^2\theta/2)}{4a\cos^4\theta/2}$$

Substituting above into equ. (47), we get

$$d^2u/d\theta^2 + u = -m/l^2\,u^2 f(1/u),$$

one gets

$$\frac{(1+2\sin^2\theta/2)}{4a\cos^4\theta/2} + \frac{1}{2a\cos^2\theta/2} = -m/l^2\,a^2 \times 4\cos^4\theta/2\,f(1/u)$$

or $$\frac{3}{4a\cos^4\theta/2} = -m/l^2\, a^2 \times 4\cos^4\theta/2\, f(1/u)$$

or $$f(1/u) = -l^2/m \times \frac{3}{16a^2\cos^8\theta/2}$$

or $$f(1/u) = -l^2/m \times 3au^4$$

Hence $$f(r)\; \alpha\; 1/r^4$$

3.9 SCATTERING AND CROSS-SECTIONS

The scattering problem in its equivalent one-body problem is concerned with the scattering (or deflection) of particles by a centre of force (also called centre of scattering). Consider a uniform beam of particles, say of electrons or of α - particles, all of the same mass and energy incident upon a centre of force O in Fig. 3.15. This point O is the centre of mass of the two particles in the original problem. The formulation is, therefore in the centre of mass system (one in which the centre of mass is at rest).

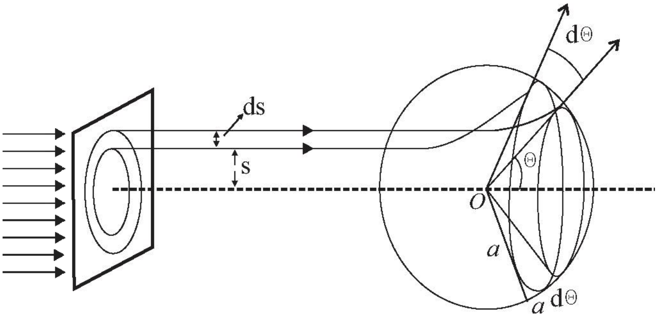

Fig. 3.15 Scattering of an incident beam of particles by a centre of force O

As a particle approaches centre of force, it would either be attracted or repelled, depending upon the nature of the field, and its orbit would deviate from the incident straight line path. After passing the centre of force, the force acting on the particle would eventually diminish (falls off to zero at very large distances) and as a result it would once again move in a straight line path. In general, the final direction of motion is not the same as the incident direction, and the particle is said to be scattered or deflected.

Let I be the intensity of incident particles (also called flux density) which represents the number of particles passing in unit time through unit area normal to the beam. The differential scattering cross-section (or cross-section for scattering) $\sigma(\Omega)$, is defined as

$$\sigma(\Omega)\, d\Omega = \frac{\text{Number of particles scattered in } d\Omega \text{ per unit time}}{\text{Incident beam intensity}} \tag{52}$$

where $d\Omega$ is an element of solid angle between Ω and $\Omega + d\Omega$ about the direction of Θ. Note that the name cross-section deserves for $\sigma(\Omega)$ as it has dimensions of area. With central forces, there being symmetry around the axis of the incident beam, the element of solid angle can be written as

$$d\Omega = \frac{\text{Surface area included between angles } \Theta \text{ and } \Theta + \quad \Theta}{}$$

$$= \frac{2\pi(a \sin \Theta) a \, d\Theta}{a^2}$$

$$= 2\,\pi \sin \Theta \, d\Theta \tag{53}$$

where Θ is the angle between the scattered and the incident directions, known as the scattering angle (see Fig. 3.15 for repulsive force field).

Let v_0 be the initial velocity of the beam (or of any incident particle in the beam) and let it be travelling in such a direction that, if undeflected (as if there is no field of force), it would pass at a distance s from the centre of force s is given the name impact parameter and is defined as the perpendicular distance between the centre of force and the incident velocity. Hence energy of the particle is given by

$$E = \tfrac{1}{2}\, m v_0^2$$

and as a result the angular momentum would be

$$l = m v_0 s = s(2\, m\, E)^{1/2} \tag{54}$$

in terms of the energy E.

The different particles in the beam have different impact parameters and are, therefore scattered through different angles Θ. The number of particles per unit time crossing a ring area $2\,\pi s ds$ (about central axis) is $(2\,\pi\, s\, ds)$ I; these particles would be scattered into $d\Omega$ (solid angle lying between Θ and $\Theta + d\Theta$). That is,

$$(2\,\pi s ds)\, I = \sigma(\Omega)\, d\Omega\, I$$

$$= -\sigma(\Theta)\, 2\pi \sin \Theta \, d\Theta \, I \tag{55}$$

The minus sign in the above expression is introduced because of the fact that an increase in impact parameter amounts to less force on the particle, resulting in a decrease in Θ. From equ. (55), the differential scattering cross-section as a function of Θ is obtained as

$$\sigma(\Theta) = -s/\sin \Theta \times ds/d\Theta \tag{56}$$

Once the differential scattering cross-section has been calculated from (56) as a function of Θ, the total scattering cross-section σ_T can be found from equs. (52) and (53) by integrating over all angles and it is given by

$$\sigma_T = \int_{4\pi} \sigma(\Omega)\, d\Omega = \int_0^{\pi} \sigma(\Theta)\, 2\pi \sin \Theta \, \mathrm{d}\Theta \tag{57}$$

3.10 SCATTERING OF CHARGED PARTICLES BY COULOMB FIELD-RUTHERFORD FORMULA

One of the most important applications of the above derived formulae is to the scattering of charged particles by Coulomb field which in this case is produced by a fixed charge -Ze (placed at the centre of force) acting on the incident particles having a charge -Z'e. This inverse-square repulsive force between the charges is

$$f = ZZ'e^2/r^2, \tag{58}$$

In analogy with our earlier treatment (See. 36, equ. 31), the force constant in the present case can be written as

$$k = -ZZ'e^2 \tag{59}$$

and the orbit equation becomes (see equ. 36),

$$1/r = -\frac{mZ\,Z'e^2}{l^2}\left[1+\sqrt{1+2El^2/\mu(-ZZ'e^2)^2}\,\cos(\theta-\theta')\right]$$

$$= -\frac{mZ\,Z'e^2}{l^2}\left[1+\sqrt{1+(2Es/ZZ'e^2)^2}\,\cos(\theta-\theta')\right] \tag{60}$$

where the value of angular momentum l has been substituted from equ. (54). The eccentricity in this case would be given by

$$\in = \sqrt{1+(2Es/ZZ'e^2)^2} \tag{61}$$

which suggests that $\in > 1$; hence the trajectory in this case would be a hyperbola as shown in Fig. 3.16. This is the negative branch trajectory of the hyperbola. The repulsive force centre is at F.

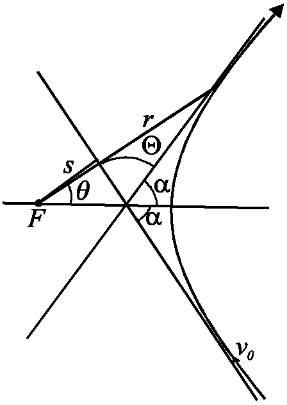

Fig. 3.16 Hyperbolic orbit in a repulsive inverse-square law of force. The centre of force being at F

If θ' in equ. (60) is chosen to be π, the perihelion corresponds to $\theta = 0$ and the equation of orbit becomes

$$1/r = mZZ'e^2/l^2\,(\in \cos\theta - 1) \qquad (60')$$

The above hyperbolic orbit equation has the same form as the elliptic orbit equ. (36) except for the change in sign:

As the path of particle in a central force field is symmetric about a line from the centre of force to the nearest point in the orbit, the two asymptotes to the orbit make equal angles with this line. The scattering angle as the particle passes the centre of force is seen from Fig. 3.16 to be

$$\Theta = \pi - 2\alpha \qquad (62)$$

Therefore,

$$\cot\Theta/2 = \cot(\pi/2 - \alpha) = \tan\alpha \qquad (63)$$

For $$r \to \infty, \quad \alpha \approx 0 \qquad (64)$$

Under the above condition, from equ. (60′)

$$\in \cos\theta - 1 = 0$$

or $$\cos\theta = \cos\alpha = 1/\in$$

As a result, from equ. (63), one gets

$$\begin{aligned}\cot^2\Theta/2 &= \tan^2\alpha = \sec^2\alpha - 1\\ &= \in^2 - 1\end{aligned}$$

and using equ. (61),

$$\cot\Theta/2 = 2Es/ZZ'e^2$$

The impact parameter as a function of scattering angle Θ, is therefore

$$s = ZZ'e^2/2E\cot\Theta/2 \qquad (65)$$

With the above result, the differential scattering cross-section using equ. (56), is given by

$$\begin{aligned}\sigma(\Theta) &= -ZZ'e^2/2E\frac{\cot\Theta/2}{\sin\Theta}ZZ'e^2/2E\,(-\mathrm{cosec}^2\,\Theta/2) \times \tfrac{1}{2}\\ &= \tfrac{1}{4}\,(ZZ'e^2/2E)^2\,1/\sin^4\Theta/2 \qquad (66)\end{aligned}$$

This gives the famous Rutherford scattering cross-section, originally derived by Rutherford for the scattering of α-particles by atomic nuclei. It may be noted that quantum mechanics in the non-relativistic limit yields the identical result for the differential scattering cross-section. If this were not so, then the Rutherford's scattering experiment which established the model of the nuclear atom would not have been properly interpretted and the history of atomic physics might have been quite different.

Note that though in atomic physics, the concept of total scattering cross-section σ_T, defined by equ. (57) is of considerable importance, but it is not always so, for sometimes it diverges. The Coulomb repulsion of like charges (Coulomb scattering) is an example. If we attempt to calculate σ_T by substituting equ. (66) in its definition, we obtain an infinite result. The physical reason behind this behaviour is not difficult to understand. From its definition σ_T is the number of particles scattered in all directions per unit time for unit incident intensity. The Coulomb field (Coulomb $1/r$ potential) being long-range force, its effects extend to infinity. The very small deflections occur only for particles with large s. Hence all particles in an incident beam of infinite lateral extent would be scattered to some extent and must be included in σ_T, thereby causing σ_T to be infinite. Thus σ_T is meaningful only when the force field cuts off sharply with distance.

3.11 RELATION BETWEEN CROSS-SECTIONS IN COM AND LABORATORY SYSTEMS

So far we have considered the scattering problems in the com system. Let us now consider the transformation of the scattering cross-section from the com system to the laboratory system. The connection between the two forms can be obtained from the fact that the number of particles scattered into a given element of solid angle must be the same in both the systems, that is, whether we measure the event in terms of θ_1 (the angle of scattering in laboratory system) or in terms of Θ (the angle of scattering in com system). That is,

$$2\pi\, \sigma(\Theta) \sin\Theta\, d\Theta\, I = 2\pi\, \sigma(\theta_1) \sin\theta_1\, d\theta_1\, I$$

or
$$\sigma(\theta_1) = \sigma(\Theta) \frac{\sin\Theta d\Theta}{\sin\theta_1 d\theta_1} = \sigma(\Theta) \frac{d(\cos\Theta)}{d(\cos\theta_1)} \tag{67}$$

where $\sigma(\theta_1)$ and $\sigma(\Theta)$ are the differential scattering cross-sections in the laboratory and com systems, respectively. In principle, (A.7) can be solved for Θ in the terms of θ_1, and hence $\sigma(\theta_1)$ evaluated in terms of $\sigma(\Theta)$, but this is quite a tedious job for arbitrary m_1/m_2 ratio. For the scattering of α-particles by nuclei (Rutherford experiment), m_1/m_2 in equ. (A.7) is very small, because m_1 is 4 atomic units whereas m_2 is approximately 100 atomic units. As a result $\theta_1 = \Theta$ or in other words $\sigma(\theta_1) = \sigma(\Theta)$. But if the masses of the two particles are equal, equ. (A.7) then becomes

$$\tan\theta_1 = \frac{\sin\Theta}{\cos\Theta + 1} = \tan\Theta/2$$

or
$$\theta_1 = \Theta/2 \tag{68}$$

As we know that Θ can have any value between 0 and π, therefore, the maximum value which θ_1 can have is 90°. That means, with equal masses the maximum value of the scattering angle in the laboratory system is 90°. That is, all the scattering is in the forward hemisphere and there can be no scattering back of that. Consequently, the differential scattering cross-section using equs. (67) and (68) is given by

$$\sigma(\theta_1) = \sigma(2\theta_1) \frac{d(\cos 2\theta_1)}{d(\cos\theta_1)}$$

$$= \sigma(2\theta_1)\frac{-\sin 2\theta_1 \times 2d\theta_1}{-\sin\theta_1 \times d\theta_1}$$

$$= \sigma(2\theta_1)\ 4\cos\theta_1,\ \theta_1 \leq \pi/2 \tag{69}$$

which shows that even if the scattering is isotropic in terms of Θ, that is, $\sigma(\Theta)$ $(=\sigma(2\theta_1))$ is constant, independent of Θ, the differential scattering cross-section $\sigma(\theta_1)$ varies as the cosine of θ_1.

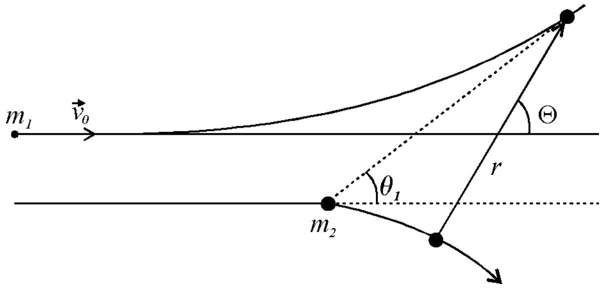

Fig. 3.17 Scattering by a target of mass $m_2 \approx m_1$, the mass of the incident particle in the laboratory system.

It is necessary to mention here that in the problems discussed earlier, it was assumed that the target was heavy as compared to the incident particle and hence was assumed to be at rest during the collision. In case the target nucleus is not heavy, the nucleus itself would move during the collision, as shown in Fig. 3.17- the masses of incident and target particle are almost equal, $m_1 \cong m_2$. The difficulty can be overcomed by considering the collision in the com system and in the final result replacing m by the reduced mass μ. θ_1 of course has already been replaced by Θ.

PROBLEMS

3.1 Two particles move about each other in circular orbits under the influence of gravitational forces, with period τ. Their motion is suddenly stopped at a given instant of time, and they are then released and allowed to fall into each other. Prove that they collide after a time $\tau/4\sqrt{2}$.

3.2 Show that if a particle describes a circular orbit under the influence of an attractive central force directed toward a point on the circle, then the force varies as the inverse-fifth power of the distance.

3.3 Evaluate approximately the ratio of mass of the Sun to that of earth, using only the length of year and of the lunar month (27.3 days), and the mean radii of earth's orbit (1.49×10^8 km) and that of the Moon's orbit (3.8×10^5 km).

3.4 Find the differential scattering cross-section for scattering of particles from a perfectly rigid sphere of radius a. That is, when the particle is moving in a force field of potential

$$V(r) = \infty, \quad r < a$$
$$= 0, \quad r > a$$

Finally comment on the total scattering cross-section.

3.5 Find the differential scattering cross-section for scattering by a spherical potential well of radius a and depth V_0, defined by the potential

$$V = 0, \quad r > a$$
$$= -V_0, \quad r \leq a$$

show that the total scattering cross-section has the expected value of πa^2.

3.6 Find the differential scattering cross-section for scattering by an inverse cube repulsive force.

3.7 Figure 3.22 represents an ellipse and the circumscribed circle of radius a. Show that

$$r = a(1 - \in \cos \alpha).$$

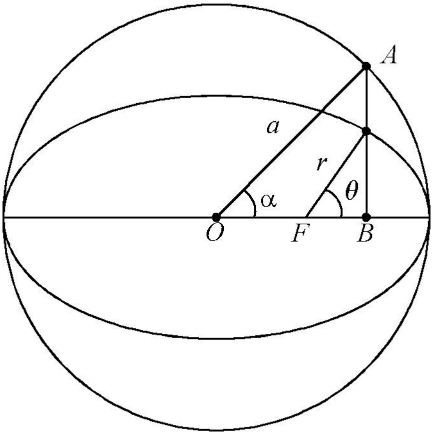

Fig. 3.22 Ellipse and circumscribed circle of radius a.

3.8 A mass m is moving around a large mass M in a circular orbit of radius a under the gravitational attraction between M and m. Suddenly the direction of the velocity of m is rotated through 90° so that it points directly at M. If mass m takes time T_1 to reach M and if the period of the original circular orbit was T_2, evaluate the ratio T_1/T_2 by using the conservation of energy and the Kepler's third law.

3.9 Show that the elliptical orbit equ. (36) can be written as

$$r = \frac{a(1 - \in)}{1 + \in \cos(\theta - \theta')}$$

where terms have their usual meaning.

SOLUTIONS

3.1 The equation of motion of equivalent one body problem, in this case, is given by

$$\mu\, d^2 r/dt^2 = -Gm_1 m_2/r^2$$

or

$$d^2r/dt^2 = -G(m_1 + m_2)/r^2$$

where m_1 and m_2 are the masses of two particles, μ their reduced mass and $r = |\vec{r}_1 - \vec{r}_2|$ is the separation between the particles. G being the gravitational constant.

Multiplying both sides of the above equation by $2dr/dt$ and then integrating, we get

$$(dr/dt)^2 = 2G(m_1 + m_2)/r + C \text{ (a constant of integration)} \quad (70)$$

Applying the initial condition, that is, at $r = a$, $dr/dt = 0$, one gets

$$C = 2G(m_1 + m_2)/a$$

Substituting above into equ. (70), results in

$$(dr/dt)^2 = 2G(m_1 + m_2)\,(1/r - 1/a)$$

or

$$dr/dt = -[2G(m_1 + m_2)(1/r - 1/a)]^{1/2}$$

where we have taken the (–)ve root as r decreases with time.

The time in which m_2 collides m_1 can be found by integrating the above expression and the result is

$$t = -\sqrt{\frac{a}{2G(m_1+m_2)}} \int_a^0 \sqrt{\frac{r}{a-r}}\, dr$$

Making the substitution, $r = a\cos^2\theta$ or $dr = -2a\cos\theta \sin\theta\, d\theta$, the above expression reduces to

$$t = -\sqrt{\frac{a}{2G(m_1+m_2)}} \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta} \times -2a\cos\theta \sin\theta\, d\theta$$

$$= \pi a^{3/2}/2\sqrt{2G(m_1+m_2)} \quad (71)$$

Since τ is given by

$$\tau = 2\pi a^{3/2}/\sqrt{G(m_1+m_2)}$$

the expression (71) reduces to

$$t = \tau/4\sqrt{2}$$

3.2 The equation of a circle with pole on its circumference (see Fig. 3.18), is given by

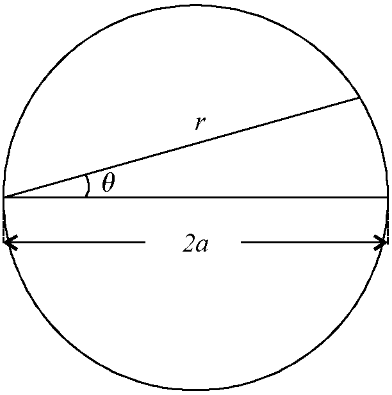

Fig. 3.18 Circle with pole on its circumference.

$$r = 2a\cos\theta \tag{72}$$

Putting $u = 1/r = \sec\theta/2a$, and hence

$$du/d\theta = \sec\theta\tan\theta/2a,$$

and

$$\begin{aligned} d^2u/d\theta^2 &= (\sec\theta\sec^2\theta + \tan\theta\sec\theta\tan\theta)/2a \\ &= [\sec^3\theta + \sec\theta(\sec^2\theta - 1)]/2a \\ &= (2\sec^3\theta - \sec\theta)/2a \end{aligned}$$

Substituting above in the equation for orbit, we get

$$d^2u/d\theta^2 + u = (2\sec^3\theta - \sec\theta)/2a + \sec\theta\tan\theta/2a = -m/l^2u^2f(1/u)$$

or $\sec^3\theta/a = 8a^2u^3 = -m/l^2u^2f(1/u)$

or $f(1/u)\ \alpha\ u^5$

which amount to $f(r)\ \alpha\ 1/r^5$

3.3 The time period of earth rotation about the sun, is given by

$$\tau_1 = 2\pi a_1^{3/2}\sqrt{1/G(m_s + m_E)} \tag{73}$$

where m_s and m_E are the masses of sun and earth, respectively. a_1 is the mean radius of the earth's orbit.

The time period of moon rotation about the earth, is given by

$$\tau_2 = 2\pi a_2^{3/2}\sqrt{1/G(m_E + m_m)} \tag{74}$$

where m_m is the mass of moon and a_2 is the mean radius of moon's orbit.

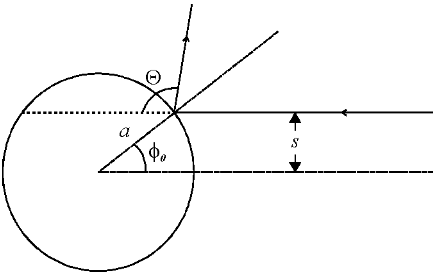

Fig. 3.19 Scattering of a particle from a rigid sphere.

From equs. (73) and (74), one gets

$$\tau_2^2/\tau_1^2 \times a_1^3/a_2^3 = (m_s + m_E)/(m_E + m_m)$$

$$= (m_s/m_E + 1)/(m_m/m_E + 1)$$

Since $m_m/m_E << 1$ and also $m_s/m_E >> 1$, from above equation, one gets

$$m_s/m_E = \tau_2^2/\tau_1^2 \times a_1^3/a_2^3$$

$$= (27.3)^2/(365)^2 \times (1.49 \times 10^8)^3/(3.8 \times 10^5)^3$$

$$\approx 3.4 \times 10^5$$

3.4 The particle in this case moves freely outside the sphere and can not penetrate into it because of the fact that $V(r) = \infty$, $r < a$. Therefore, the path of the particle consists of two straight lines symmetrical about the radius to the point where the particle strikes the sphere (see Fig. 3.19).

The impact parameter s is given by

$$s = a \sin \phi_0 = a \sin(\pi - \Theta)/2$$

$$= a \cos \Theta/2$$

As a result, the differential scattering cross-section (equ. 56), is given by

$$\sigma(\Theta) = -s/\sin\Theta \times ds/d\Theta$$

$$= -\frac{s}{2\sin\Theta/2\; cis\,\Theta/2} \times -a\sin\Theta/2\ ½$$

$$= ¼\, a^2$$

that is, the scattering is isotropic in the C-system.

On integrating $\sigma(\Theta)$ over the total solid angle, the total scattering cross-section.

$$\sigma = \int_{4\pi} \sigma(\Omega)d\Omega = ¼\, a^2 \times 4\pi = \pi\, a^2,$$

which is nothing but the cross-sectional area of the sphere. This is an expected result.

3.5 The angle of incidence in the present case is α and the angle of refraction is β (see Fig. 3.20). Therefore, the index of refraction.

$$n = \sin \alpha/\sin \beta$$

The angle of scattering (see Fig. 3.20) is

$$\Theta = 2(\alpha - \beta)$$

Therefore, $\sin \beta / \sin \alpha = \sin(\alpha - \Theta/2)/\sin\alpha$

$$= \cos \Theta/2 - \cot \alpha \sin \Theta/2$$

$$= 1/n \tag{75}$$

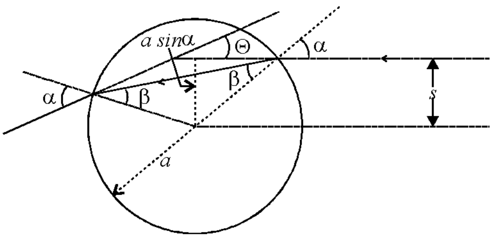

Fig. 3.20 Scattering by a spherical potential well.

The impact parameter s (see Fig. 3.20), is

$$s = a \sin \alpha \tag{76}$$

From equs. (75) and (76), one gets

$$\cos\Theta/2 - \frac{\cos\alpha}{\sin\alpha}\sin\Theta/2 = 1/n$$

or $\cos \Theta/2 - (1 - s^2/a^2)^{1/2}/s/a \times \sin\Theta/2 = 1/n$

$$\text{or} \qquad (1 - s^2/a^2) \,/\, (s/a)^2 = \frac{\cos\Theta/2 - 1/n}{\sin\Theta/2}$$

$$\text{or} \qquad s^2 = a^2 (n^2 \sin^2 \Theta/2)/(n^2 + 1 - 2\, n \cos \Theta/2) \tag{77}$$

Differentiating above w.r.t. Θ, one gets

$$2s\,ds/d\Theta = a^2 \frac{\begin{array}{c}(n^2+1-2n\cos\Theta/2)n^2 \times 2\sin\Theta/2\cos\Theta/2 \times \\ 1/2 - n^2\sin^2\Theta/2 \times 2n\sin\Theta/2\;1/2\end{array}}{(n^2+1-2n\cos\Theta/2)^2}$$

$$= \frac{\begin{array}{c}a^2n^2\sin\Theta/2 \times (n^2+1-2n\cos\Theta/2) \\ \cos\Theta/2 - n(1-\cos^2\Theta/2)\end{array}}{(n^2+1-2n\cos\Theta/2)^2}$$

Therefore, the differential scattering cross-section,

$$\sigma(\Theta) = s/\sin\Theta \;\; |ds/d\Theta|$$

$$= \frac{n^2a^2}{4\cos\Theta/2}\,\frac{(n\cos\Theta/2-1)(n-\cos\Theta/2)}{(n^2+1-2n\cos\Theta/2)^2}$$

The impact parameters varies from 0 to a (for $s > a$, the particles pass without deviation). As a result the scattering angle (Θ) takes values from 0 to $\Theta_{\max}$, given by

$$\cos \Theta_{\max}/2 = 1/n$$

Therefore, the total scattering cross - section (taking $z = \cos \Theta/2$, $dz = -\sin\Theta/2 \times ½\, d\Theta$),

$$\sigma_T = 2\pi \int_0^{\Theta_{\max}} \sigma(\Theta)\sin\Theta \; d\Theta = 2\pi n^2a^2 \int_0^1 \frac{(nz-1)(n-z)}{(n^2-2nz+1)^2}\,dz$$

The result of integration is πa^2, as expected.

3.6 Since the force field obeys the inverse cube law,

$$f = k/r^3 = k\,u^3,\; u = 1/r \tag{75}$$

Substituting (75) into (47), one gets

$$\frac{d^2u}{d\theta^2} + u = -m\,ku/l^2$$

or $$\frac{d^2u}{d\theta^2} + (1 + mk/l^2)u = 0$$

which has the general solution

$$u = 1/r = A\sin(1 + mk/l^2)^{1/2}\,\theta + B\cos\,(1 + mk/l^2)^{½}\,\theta$$

where A and B are constants. The above equation can further be written in the form

$$1/r = C\sin\{(1 + mk/l^2)^{1/2}\,(\theta - \theta')$$

where the constants C and θ' are related to A and B. The above equation can be written as

$$1/r = C\sin\alpha,\; \alpha = (1 + mk/l^2)^{1/2}\,(\theta - \theta') \tag{76}$$

Note that $r_{\min}$ occurs for $\alpha = \pi/2$ whereas $r = \infty$ corresponds to $\alpha = 0$ or π (see Fig. 3.21). We see that

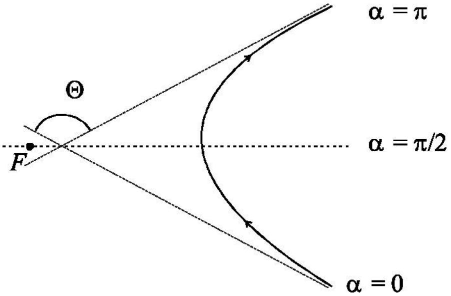

Fig. 3.21 Scattering by inverse-cube law of force. The centre of force being at F.

$$\Theta = \pi - 2[\theta(\alpha = \pi/2) - \theta(\alpha = 0)]$$

$$= \pi - 2\left[\frac{\pi}{2(1 + mk/1^2)^{½}} + \theta' - \theta'\right]$$

where use of equ. (76) has been made.

Further putting $l = s(2\text{mE})^{1/2}$ into above, we get

$$\Theta = \pi[1 - \sqrt{2Es^2/(2Es^2 + k)}\,] \tag{77}$$

Differentiating above with respect to s, we get

$$\frac{d\Theta}{ds} = -\pi k\sqrt{2E}/(2Es^2 + k)^{3/2}$$

Further using the relation (56), we get

$$\sigma(\Theta) = -\frac{s}{\sin\Theta}\times\frac{1}{d\Theta/ds} = \frac{s}{\sin\Theta}\frac{(2Es^2+k)^{3/2}}{\pi k\sqrt{2E}} \qquad (78)$$

From equ. (77)

$$\sqrt{2Es^2/(2Es^2+k)} = \left(\frac{\pi-\Theta}{\pi}\right)^2$$

On squaring both the sides, and then simplifying, we get

$$s = \sqrt{\frac{k}{2E\Theta(2\pi-\Theta)}}(\pi-\Theta)$$

Putting above in equ. (78), gives

$$\sigma(\Theta) = \frac{\pi^2 k}{2E}\frac{(\pi-\Theta)}{\Theta^2\sin\Theta(2\pi-\Theta)^2}$$

3.7 From Fig. 3.22,

$$OB = AO \text{ Cos } \alpha$$

or $$(OF + FB) = a\cos\alpha$$

or $$a\in + r\cos\theta = a\cos\alpha \qquad (79)$$

The equation of ellipse referred to focus (F) as a pole is given by

$$r = a(1-\in^2)/(1+\in\cos\theta)$$

or $$r\in\cos\theta + a\in^2 = a - r \qquad (80)$$

Multiplying (79) by $\in$, we get

$$r\in\cos\theta + a\in^2 = a\in\cos\alpha$$

Putting above into equ. (80), we get

$$a - r = a\in\cos\alpha$$

or $$r = a(1-\in\cos\alpha)$$

3.8 Since the mass m is moving in a circle of radius a with a velocity v_0 under the gravitational attraction between M and m, therefore

$$mv_0^2/a = GMm/a^2$$

or $$v_0^2 = GM/a \qquad (81)$$

Let v be the velocity of mass m at a distance $r(< a)$ from the centre (the mass M), then from the conservation of energy

$$\tfrac{1}{2}mv_0^2 - \frac{GMm}{a} = \tfrac{1}{2}mv^2 - \frac{GMm}{r} \qquad (82)$$

Putting the value of υ_0^2 from equ. (81) into equ. (82), we get

$$v^2 = GM(2/r - 1/a) \tag{83}$$

Further $$T_1 = \int_a^0 -dr/v$$

as directions of r and v are opposite ($dr/dt = -v$). Making use of equ. (83), we get

$$T_1 = \int_0^a dr/v = \int_0^a 1/(GM)^{1/2}\, dr/(2/r - 1/a)^{1/2}$$

$$= (1/GM)^{1/2} \int_0^a a^{3/2}\left(\frac{x}{2-x}\right)^{1/2} dx$$

where we have put $r = ax$. Further putting

$$x = 1 + \cos\theta,\ dx = -\sin\theta\, d\theta,$$

we get

$$T_1 = (a^3/GM)^{1/2} \int_{\pi/2}^{\pi} \sin\theta\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/2} d\theta$$

$$= (a^3/GM)^{1/2} \int_{\pi/2}^{\pi} \left(1-\cos^2\theta\right)^{1/2}\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/2} d\theta$$

$$= (a^3/GM)^{1/2} \int_{\pi/2}^{\pi} (1 + \cos\theta)d\theta$$

$$= (\pi/2 - 1)\,(a^3/GM)^{1/2}$$

From Kepler's third law,

$$T_2 = 2\,\pi(a^3/GM)^{1/2}$$

Therefore

$$T_1/T_2 = \frac{\pi - 2}{4\pi}$$

3.9 We have already shown that

$$a = -k/2E \tag{41}$$

Further from equ. (38),

$$\in^2 = 1 + 2El^2/\mu k^2$$

or $$1 - \in^2 = -2El^2/\mu k^2 = l^2/\mu ka$$

where we have used equs. (41). Therefore,

$$l^2/\mu k = a(1 - \in^2)$$

In terms of the above, the elliptical orbit equ. (36) can be written as

$$r = \frac{a(1-\in^2)}{(1+\in\cos(\theta-\theta')}$$

4
CHAPTER

Motion of a Rigid Body

4.1 DESCRIPTION OF A RIGID BODY

A rigid body is defined as a system of particles in which the interparticle distances remain constant, that is, do not vary with time even when the body is in motion or is under the action of external forces. This is an idealized definition of a rigid body as it does become deformed under the action of applied forces. However, the majority of solid bodies change so little in shape and size under ordinary conditions that these changes may be neglected in describing the motion of the body as a whole.

To describe the configuration or position in space of rigid body, we use two systems of coordinates (see Fig. 4.1). (i) a fixed (or inertial) coordinate system xyz given the name space set of axes and (ii) a moving coordinate system $x'y'z'$ given the name body set of axes supposed to be fixed in the rigid body. The origin of the moving coordinate system is taken as the com of the rigid body.

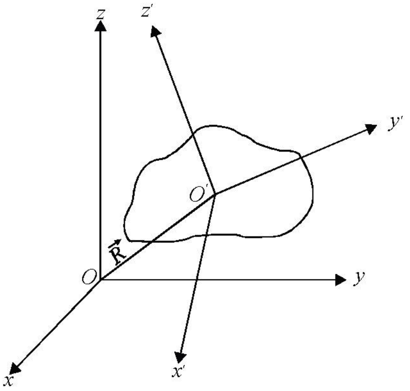

Fig. 4.1 The configuration of a rigid body is specified by six independent coordinates, three components of the radius vector $\vec{R}$ (the coordinates of O' w.r.t. O) and the three angles giving the orientation of the $x'y'z'$ axes (moving coordinate system or the body set of axes) relative to the xyz axes (fixed or inertial coordinate system or space set of axes).

The position of the rigid body w.r.t. the fixed coordinate system is completely determined if the position of the moving coordinate system is specified. The $\vec{R}$ be the radius vector of the origin O' of the moving coordinate system (see Fig. 4.1). The orientation of the body set of axes (primed axes) relative to the space set of axes (unprimed axes) can be specified with the help of three angles which together with the three components of $\vec{R}$ make six coordinates to completely specify the position of the rigid body corresponding to the translational and rotational motions.

4.2 ANGULAR MOMENTUM

Consider an infinitesimal rotation of a rigid body about an axis Oz (or z') which corresponds to the fact that any point P (or particle) of the body moves anticlockwise in a circle with O' (com of the body) as the centre (See Fig. 4.2). We will define $d\vec{\phi}$ as a vector of infinitesimal rotation having magnitude $d\phi$ and direction along the axis of rotation.

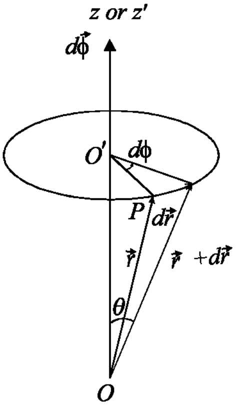

Fig. 4.2 Rotation of a rigid body about an axis through fixed point O.

The resulting increment $d\vec{r}$ in the radius vector $\vec{r}$ in the system undergoing infinitesimal rotation about an axis through fixed point O is related to the angle by $|d\vec{r}| = r \sin\theta \, d\phi$ (see Fig. 4.2), where $r \sin\theta$ is the radius of the circle in which point P moves. The direction of $d\vec{r}$ is perpendicular to the plane containing and $d\vec{\phi}$. Therefore,

$$d\vec{r} = d\vec{\phi} \times \vec{r} \tag{1}$$

Dividing the above equation by dt, the infinitesimal time interval during which the rotation occurs, one obtains, the linear velocity of point P at position $\vec{r}$ in body or relative to the fixed system of coordinates, given by

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r} \tag{2}$$

in which

$$\vec{\omega} = d\vec{\phi}/dt$$

is called the angular velocity of rotation and is directed along the axis of rotation, Oz.

The total angular momentum of the body with respect to the fixed point* is the sum of angular momenta of individual particles and is given by

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i m_i \left(\vec{r}_i \times \vec{\upsilon}_i\right) \tag{3}$$

As a result, equ. (3) becomes

$$\vec{L} = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

where $\vec{r}_i$ and $\vec{\upsilon}_i$ are the radius and velocity vectors, respectively of the *i*th particle relative to the fixed point O and the summation is carried out over all the particles of the rigid body. As the linear $\vec{\upsilon}_i$ w.r.t. the fixed system of coordinates arises solely from the rotational motion of the rigid body about the fixed point is therefore, given by

$$\vec{\upsilon}_i = \vec{\omega} \times \vec{r}_i$$

Substituting above into equ.(3) and using the identity for the vector triple product

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$$

the expression for $\vec{L}$ can be written as

$$\vec{L} = \Sigma m_i \quad \left(\vec{r}_i \cdot \vec{r}_i\right) - r_i\left(\vec{r}_i \cdot \quad\right) = \sum_i m_i \left\{\vec{\omega} r_i^2 - \vec{r}_i\left(\vec{r}_i \cdot \vec{\omega}\right)\right\} \tag{4}$$

Thus the *x*-component of the angular momentum becomes

$$\begin{aligned} L_x &= \sum_i m_i\{\omega_x r_i^2 - x_i\,(x_i\omega_x + y_i\omega_y + z_i\omega_z)\} \\ &= \sum_i \omega_x m_i(r_i^2 - x_i^2) - \sum_i \omega_y m_i x_i y_i - \sum_i \omega_z m_i x_i z_i \end{aligned} \tag{5}$$

with similar expressions for the other components of $\vec{L}$. These expressions imply that each of the components of the angular momentum is a linear function of all the components of the angular velocity or in other words the angular momentum is related to the angular velocity by a linear transformation. We may, therefore write

$$\begin{aligned} L_x &= I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z, \\ L_y &= I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z, \\ L_z &= I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z \end{aligned} \tag{6}$$

where

$$\begin{aligned} I_{xx} &= \sum_i m_i\{r_i^2 - x_i^2) = \sum_i m_i\,(y_i^2 + z_i^2), \\ I_{yy} &= \sum_i m_i\{z_i^2 + x_i^2), \\ I_{zz} &= \sum_i m_i\{x_i^2 + y_i^2), \\ I_{xy} &= I_{yx} = -\sum_i m_i x_i y_i, \end{aligned} \tag{7}$$

---

* It the axis of rotation lies within the body, then any point along the axis will be at rest in the laboratory frame as well as in the body frame and will constitute a convenient choice of the reference point.

$$I_{xz} = I_{zx} = -\sum_i m_i x_i z_i,$$

$$I_{yz} = I_{zy} = -\sum_i m_i y_i z_i,$$

In matrix notation, equs. (6) are expressed by

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad (8)$$

The nine coefficients $I_{xx,}$ I_{xy}, etc. are thus the nine elements of the transformation matrix. The diagonal elements I_{xx}, I_{yy} and I_{zz} are known as the moments of inertia about. x, y and z axes, respectively and the off diagonal elements I_{xy}, I_{xz}, I_{yz}, etc. are designated as products of inertia.

Note that the coefficients (or matrix elements) appear in equs. (7) in the form which is suitable if the rigid body is composed of discrete particles. For continuous bodies one has to replace the summation by a volume integration and mass by mass density $\rho(\vec{r})$. As a result, the moments of inertia and the products of inertia can be express as

$$I_{xx} = \int_V \rho(\vec{r})\,(r^2 - x^2)dV, \qquad (9)$$

$$I_{xy} = -\int_V \rho(\vec{r})\,xy\,dV, \qquad (10)$$

and so on. The integration is to be carried out over the entire volume of the body.

It may further be noted that so far we have not specified the coordinate system in resolving the components of $\vec{L}$. But from now onwards, we would take it to be a system fixed in the body which was earlier (sec. 4.1) denoted by primes and was called body set of axes. The various distances x_i, y_i, z_i are then constant in time so that the matrix elements are likewise constant in time. Unless otherwise specified all coordinates used for rest of the chapter would refer to the body set of axes or coordinate system fixed in the rigid body.

4.3 THE INERTIA TENSOR

The equ. (8) may be considered as the matrix representation of the operator equation.

$$\vec{L} = I\vec{\omega} \qquad (11)$$

where the symbol I stands for the operator whose matrix elements are the inertia coefficients appearing in equ. (8). Note that I is not a scalar because $\vec{L}$ and $\vec{\omega}$ are two physically different vectors having different dimensions. Equ. (11) is to be read as saying that the operator I acting on the vector $\vec{\omega}$ results in a physically new vector $\vec{L}$. Since I have nine components, it is a tensor of rank two or of second rank and is termed as moment of inertia (or inertia) tensor. Further it is symmetrical as $I_{xy} = I_{yx}$ and so on. As a result it has only six independent components.

4.4 ROTATIONAL KINETIC ENERGY OF A RIGID BODY

The expression for the rotational kinetic energy of a rigid body can be obtained from the relation

$$T = \sum_i \tfrac{1}{2} m_i v_i^2$$

where v_i is the velocity of the ith particle relative to the fixed point.

Using the relation,

$$\vec{v}_i = \quad \times \vec{r}_i$$

where $\vec{r}_i$ is the radius vector of ith particle relative to the fixed point, one gets

$$\begin{aligned} \mathrm{T} &= \sum_i \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} \sum_i m_i \vec{v}_i \cdot \vec{v}_i \\ &= \tfrac{1}{2} \sum_i m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i) \\ &= \tfrac{1}{2} \sum_i m_i \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i) \\ &= \tfrac{1}{2} \vec{\omega} \cdot \vec{L} \end{aligned} \qquad (12)$$

where we have used the vector identity, $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{A} \times \vec{C}\right)$

4.5 PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA

Let us choose a set of body axes such that the off-diagonal elements, that is, I_{xy}, I_{yz} and I_{zx} appearing in the inertia tensor vanish or in other words the inertia tensor becomes diagonal†. The set of axes so chosen are called the principal axes and the corresponding moments of inertia (or the diagonal elements of the inertia tensor) as the principal moments of inertia. We shall denote them by I_1, I_2 and I_3.

As a result, equ. (8) becomes

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} I_1\omega_x + 0 + 0 \\ 0 + I_2\omega_y + 0 \\ 0 + 0 + I_3\omega_z \end{bmatrix}$$

which amounts to

$$L_x = I_1\omega_x, \quad L_y = I_2\omega_y, \quad L_z = I_z\omega_z \qquad (13)$$

and similarly

$$\begin{aligned} T &= \tfrac{1}{2} \vec{\omega} \cdot \vec{L} = \tfrac{1}{2}(\omega_x L_x + \omega_y L_y + \omega_z L_z) \\ &= \tfrac{1}{2}(I_1 \omega_x^2 + I_2 \omega_y^2 + I_3 \omega_z^2) \end{aligned} \qquad (14)$$

where we have substituted the values of L_x, L_y and L_z from equ. (13).

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† Any symmetrical tensor of rank two can be reduced to a diagonal form.

4.6 DETERMINATION OF THE PRINCIPAL AXES

We would now show that even if the inertia tensor is not symmetrical, one can find out the principal moments of inertia by diagonalizing the inertia matrix. We make use of the fact that if the axis of rotation coincides with the principal axis, then both the angular momentum $\vec{L}$ and the angular velocity $\vec{\omega}$ are directed along this axis. If I is the moment of inertia about this axis, we may write.

$$\vec{L} = I\vec{\omega} \tag{15}$$

Comparing equs. (15) and (6), the components of $\vec{L}$ can be written as

$$\begin{aligned} L_x &= I\,\omega_x = I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z, \\ L_y &= I\,\omega_y = I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z, \\ L_z &= I\,\omega_z = I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z \end{aligned} \tag{16}$$

After rearranging the terms the above equation can be written as

$$\begin{aligned} (I_{xx} - I)\,\omega_x + I_{xy}\,\omega_y + I_{xy}\omega_z &= 0, \\ I_{yx}\,\omega_x + (I_{yy} - I)\,\omega_y + I_{yz}\,\omega_z &= 0, \\ I_{zx}\,\omega_x + I_{zy}\,\omega_y + (I_{zz} - I)\,\omega_z &= 0 \end{aligned} \tag{17}$$

For the above equations to have nontrivial solution, the determinant of the coefficients must vanish, that is

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0 \tag{18}$$

The above equation is called a secular or characteristic equation and is cubic in I having the form

$$-I^3 + AI^2 + BI + C = 0 \tag{19}$$

where the constants A, B, and C depend on the values of the moments of inertia and the products of inertia coefficients. Each of the three roots I_1, I_2 and I_3, called principal moments of inertia, corresponds to the moments of inertia about one of the three principal axes.

The direction of any one of the principal axes can be determined by substituting for I equal to one of the three roots I_1, I_2, or I_3, say I_1 into equ. (17) and determining the ratios of the components of the angular velocity $\vec{\omega}$, that is, $\omega_1 : \omega_2 : \omega_3$. This step yields the direction cosines of the 1-axis. The directions of the principal axes corresponding to the principal moments of inertia I_2 and I_3 can be determined by the similar procedure.

In most cases of practical interest, a rigid body would have a regular shape and the principal axes can be determined intuitively from an examination of its symmetry.

For example, if the rigid body has cylindrical symmetry, one of the principal axes would coincide with the symmetry axis. The other two axes would lie in a plane perpendicular to the symmetry axis (the three axes being orthogonal). Evidently, in this particular case, if I_1 is the principal moment of inertia about the symmetry axis, then $I_2 = I_3$. If symmetry is spherical, in that case $I_1 = I_2 = I_3$.

A detailed procedure for finding the principal axes in a general case would be described through solution of the problem 4.3.

4.7 EQUATION OF MOTION OF A RIGID BODY

When a rigid body moves with one point fixed, the total orbital angular momentum of the body about that point is given by

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i \tag{20}$$

where $\vec{r}_i$ and $\vec{p}_i$ denote the radius and momentum vectors of the *ith* particle relative to the fixed point and the sum is over all the particles.

Therefore,

$$\begin{aligned} d\vec{L}/dt &= \sum_i \dot{\vec{r}}_i \times \vec{p}_i + \sum_i \vec{r}_i \times \dot{\vec{p}}_i \\ &= \sum_i \dot{\vec{r}}_i \times m_i \dot{\vec{r}}_i + \sum_i \vec{r}_i \times \vec{F}_i \\ &= 0 + \sum_i \vec{r}_i \times \vec{F}_i \\ &= \sum_i \vec{r}_i \times \vec{F}_i^e + \sum_{\substack{i,k \\ i \neq k}} \vec{r}_i + \vec{F}_{ik}^{\text{int}} \end{aligned} \tag{21}$$

in which $\vec{F}_i^e$ is the external force on the *ith* particle and $\vec{F}_{ik}^{\text{int}}$ corresponds to the internal force on the *kth* particle because of the *ith* particle. F_{ii} is of course zero.

Since the 2nd term in equ. (21) contains terms of the type

$$\vec{r}_1 \times \vec{F}_{12}^{\text{int}} + \vec{r}_2 \times \vec{F}_{21}^{\text{int}}$$

and since

$$\vec{F}_{21}^{\text{int}} = -\vec{F}_{21}^{\text{int}} \text{ from the Newton's 3rd law,}$$

$$\vec{r}_1 \times \vec{F}_{21}^{\text{int}} + \vec{r}_2 \times \vec{F}_{21}^{\text{int}} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21}^{\text{int}} = \vec{r}_{21} \times \vec{F}_{21}^{\text{int}} = 0$$

as $\vec{r}_{21}$ and $\vec{F}_{21}^{\text{int}}$ are parallel. This amounts to the fact that the total torque due to the internal forces vanishes.

As a result, the equation of motion (21) of the a rigid body reduces to

$$d\vec{L}/dt = \sum_i \dot{\vec{r}}_i \times \vec{F}_i^e = \vec{N}^e \tag{22}$$

where $\vec{N}^e$ represents the total external torque acting on body due to the external force.

4.8 RATE OF CHANGE OF A VECTOR

Let $x\,y\,z$ and $x'y'z'$ represent the space and body set of axes, that is, inertial and non-inertial (rotating) frames S and S', respectively. The origins of the two sets of axes are coincident. The unit vectors in the two coordinate systems are taken to be $\hat{x}, \hat{y}, \hat{z}$ and $\hat{x}', \hat{y}', \hat{z}'$, respectively, as shown in the Fig. 4.3.

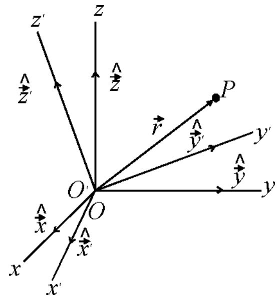


Fig. 4.3 The axes x', y' and z' (body set of axes) of system S' (called non-inertial frame) are rotating w.r.t. the axes x, y and z (space set of axes) of system S (called inertial frame).

Let the position of point P in space shown in Fig. 4.3 be represented by a vector $\vec{r}$. Since the origins of two coordinate systems coincide, $\vec{r}$ would be the same in both systems; only the components of $\vec{r}$ are different along the different axes. Thus the position vector $\vec{r}$ in terms of its components along either set of axes or coordinate system can be written as

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \tag{23}$$

$$\vec{r} = x'\,\hat{x}' + y'\,\hat{y}' + z'\,\hat{z}' \tag{24}$$

where equ. (23) represents the components of $\vec{r}$ along the space set of axes whereas equ. (24) that along the body set of axes.

From the above equations,

$$d\vec{r}/dt = \dot{x}\,\hat{x} + \dot{y}\,\hat{y} + \dot{z}\,\hat{z} \tag{25}$$

and

$$d'\vec{r}/dt = \dot{x}'\hat{x}' + \dot{y}'\hat{y}' + \dot{z}'\hat{z}' \tag{26}$$

where d/dt denotes the time derivative w.r.t. the unprimed (fixed) coordinate system and d'/dt w.r.t. the primed (rotating) coordinate system. The above equations, therefore, give the unprimed time derivative of $\vec{r}$ in terms of the unprimed components and the primed time derivative of $\vec{r}$ in terms of the primed components. The unit vectors are of course regarded as constant vectors in this situation.

One can also obtain a formula for the time derivative in primed components by taking the unprimed time derivative of equ. (24), remembering that the unit vectors $\hat{x}', \hat{y}', \hat{z}'$ are moving relative to the unprimed coordinate system and hence are functions of time. Therefore,

$$d\vec{r}/dt = \left(\dot{x}'\hat{x}' + \dot{y}'\hat{y}' + \dot{z}'\hat{z}'\right) + \left(x'\dot{\hat{x}}' + y'\dot{\hat{y}}' + z'\dot{\hat{z}}'\right)$$

$$= d'\vec{r}/dt + x'\dot{\hat{x}}' + y'\dot{\hat{y}}' + z'\dot{\hat{z}}' \qquad (27)$$

where use of equ. (26) has been made

We could now make use of equ. (2) to evaluate the last three terms in the above equation. Thus

$$d\hat{x}'/dt = \vec{\omega} \times \hat{x}', d\hat{y}'/dt = \vec{\omega} \times \hat{y}', d\hat{z}'/dt = \vec{\omega} \times \hat{z}' \qquad (28)$$

Substituting above in equ. (27), one obtains

$$d\vec{r}/dt = d'\vec{r}/dt + x'(\vec{\omega} \times \hat{x}') + y'(\vec{\omega} \times \hat{y}') + z'(\vec{\omega} \times \hat{z}')$$

$$= d'\vec{r}/dt + \vec{\omega} \times \vec{r} \qquad (29)$$

Instead of using the notations primed and unprimed coordinate systems for body set and for space set of axes, if we use the letters b and s, respectively for these, equ. (29) takes the form

$$(d\vec{r}/dt)_s = (d\vec{r}/dt)_b + \vec{\omega} \times \vec{r} \qquad (30)$$

Therefore, knowing the time derivative of $\vec{r}$ in the space set of axes (fixed coordinate system), one can find the time derivative of $\vec{r}$ in the body set of axes (rotating coordinate system) by using equ. (30). In general, one can write the relation between $(d/dt)_s$ and $(d/dt)_b$ as

$$(d/dt)_s = (d/dt)_b + \vec{\omega} \times \qquad (31)$$

4.9 EULER'S EQUATIONS OF MOTION

The angular momentum vector $\vec{L}$ can be expressed in terms of the components of the angular velocity vector $\vec{\omega}$ as

$$\vec{L} = L_{x'}\hat{x}' + L_{y'}\hat{y}' + L_{z'}\hat{z}'$$

$$= I_1\omega_{x'}\hat{x}' + I_2\omega_{y'}\hat{y}' + I_3\omega_{z'}\hat{z}' \qquad (32)$$

where I_1, I_2, and I_3 are the principal moments of inertia and $\hat{x}'$, $\hat{y}'$, $\hat{z}'$ are the unit vectors of the body coordinate system pointing along the principal axes of the body. $\omega_{x'}$, $\omega_{y'}$ and $\omega_{z'}$ are the components of $\vec{\omega}$ along the principal axes.

Remembering that the principal moments of inertia and the unit vectors along the body set of axes are constant w.r.t. the body coordinate system, the time derivative of $\vec{L}$ w.r.t. the rotating body coordinate system, from equ. (32) can be written as

$$(d\vec{L}/dt)_b = d'\vec{L}/dt = I_1\dot{\omega}_{x'}\hat{x}' + I_2\dot{\omega}_{y'}\hat{y}' + I_3\dot{\omega}_{z'}\hat{z}' \qquad (33)$$

Now from equ. (22) of the rotational motion of the rigid body in a fixed (or inertial or space) coordinate system and from equ. (31),

$$N^e = (d\vec{L}/dt)_s = (d\vec{L}/dt)_b + \vec{\omega} \times \vec{L} \tag{34}$$

Substituting the value of $(d\vec{L}/dt)_b$ from equ. (33) into equ. (34) and after removing the superscript e from N from now onwards, one gets

$$\begin{aligned}
\vec{N} &= I_1\dot{\omega}_{x'}\,\hat{x}' + I_2\,\dot{\omega}_{y'}\,\hat{y}' + I_3\,\dot{\omega}_{z'}\,\hat{z}' + \vec{\omega} \times \vec{L} \\
&= I_1 \;{}_{x'}\,\hat{x}' + I_2\dot{\omega}_{y'}\,\hat{y}' + I_3\dot{\omega}_{z'}\,\hat{z}' + (\omega_{x'}\,\hat{x}' + \omega_{y'}\,\hat{y}' \\
&\quad + \omega_{z'}\,\hat{z}') \times (I_1\omega_{x'}\,\hat{x}' + I_2\omega_{y'}\,\hat{y}' + I_3\omega_{z'}\,\hat{z}') \\
&= I_1\dot{\omega}_{x'}\,\hat{x}' + I_2\dot{\omega}_{y'}\,\hat{y}' + I_3\dot{\omega}_{z'}\,\hat{z}' + I_2\omega_{x'}\,\omega_{y'}\,\hat{z}' \\
&\quad - I_3\omega_{x'}\,\omega_{z'}\,\hat{y}' - I_1\omega_{y'}\,\omega_{x'}\,\hat{z}' + I_3\,\omega_{y'}\,\omega_{z'}\,\hat{x}' \\
&\quad + I_1\omega_{z'}\,\omega_{x'}\,\hat{y}' - I_2\omega_{z'}\,\omega_{y'}\,\hat{x}'
\end{aligned}$$

The scalar (or dot) products of the above equation with the unit vectors $\hat{x}'$, $\hat{y}'$ and $\hat{z}'$ yield the equations of motion.

$$\vec{N}\cdot\hat{x}' = N_{x'} = I_1\dot{\omega}_{x'} - (I_2 - I_3)\,\omega_{y'}\,\omega_{z'} \tag{35}$$

$$\vec{N}\cdot\hat{y}' = N_{y'} = I_2\dot{\omega}_{y'} - (I_3 - I_1)\,\omega_{z'}\,\omega_{x'} \tag{36}$$

$$\vec{N}\cdot\hat{z}' = N_{z'} = I_3\dot{\omega}_{z'} - (I_1 - I_2)\,\omega_{y'}\,\omega_{x'} \tag{37}$$

The above three equations are known as the Euler's equations of motion for the rotation of a rigid body.

4.10 TORQUE FREE MOTION OF A RIGID BODY

As an example of the application of Euler's equation of motion, we consider the motion of a rigid body not subject to any net forces or torques. The centre of mass of the body is then either at rest (or stationary) or is moving uniformly. In such a case Euler's equations reduce to

$$I_1\dot{\omega}_1 = \omega_2\,\omega_3\,(I_2 - I_3), \tag{36}$$

$$I_2\dot{\omega}_2 = \omega_3\,\omega_1\,(I_3 - I_1), \tag{37}$$

$$I_3\dot{\omega}_3 = \omega_1\,\omega_2\,(I_1 - I_2) \tag{38}$$

where we have put

$$\omega_{x'} = \omega_1,\ \omega_{y'} = \omega_2,\ \omega_{z'} = \omega_3$$

Furthermore, we would limit our discussion to the case in which the body is symmetrical. The example could be a symmetrical top in which two of the principal moments of inertia are the same. We choose the principal z'(or z)-axis as the symmetry axis so that $I_1 = I_2$. Euler's equations (36) – (38) then reduce to

$$I_1\dot{\omega}_1 = (I_1 - I_3)\,\omega_3\,\omega_2, \tag{39}$$

$$I_1\dot{\omega}_2 = -\,(I_1 - I_3)\,\omega_3\,\omega_1, \tag{40}$$

$$I_3\dot{\omega}_3 = 0 \tag{41}$$

Equ. (41) states that ω_3, the component of the angular velocity along the symmetry axis of the rigid body, is a constant. Inserting this result in the remaining two equs. (39) and (40) yields

$$\dot{\omega}_1 = -\Omega\,\omega_2, \tag{42}$$

$$\dot{\omega}_2 = \Omega\,\omega_1 \tag{43}$$

where Ω is an angular frequently given by

$$\Omega = \frac{I_3 - I_1}{I_1}\,\omega_3 \tag{44}$$

To solve for ω_1, take the time derivative of (39) and substitute the value of $\dot{\omega}_2$ from (43). Doing so one gets

$$\ddot{\omega}_1 = -\Omega\dot{\omega}_2 = -\,\Omega^2\,\omega_1 \tag{45}$$

which is the standard differential equation for simple harmonic motion with the typical solution

$$\omega_1 = A\cos\Omega\,t \tag{46}$$

where A is some constant.

The corresponding solution for ω_2 can be found by substituting the above expression for ω_1, back in equ. (42). As a result, one gets

$$\omega_2 = A\sin\Omega\,t \tag{47}$$

Squaring the two equs. (46) and (47) and then adding, one gets

$$\omega_1^2 + \omega_2^2 = A^2 \tag{48}$$

which amounts to the fact that the vector $\omega_1\hat{x}' + \omega_2\hat{y}' = \vec{A}$ has a constant magnitude and rotates about z' axis of the body with the angular velocity Ω (see Fig. 4.4). Furthermore, according to the equ. (41), ω_3 is a constant, therefore, the magnitude of the total angular velocity.

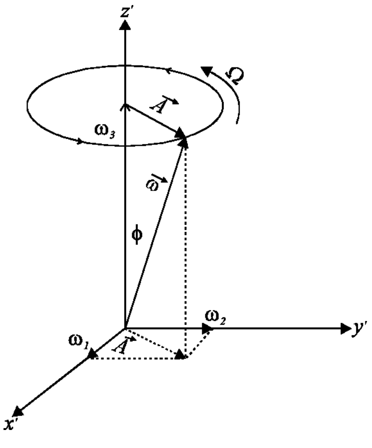


Fig. 4.4 Precession of the angular velocity vector $\vec{\omega}$ about the z'axis of symmetry in the force-free motion of a symmetrical rigid body.

$$\vec{\omega} = \omega_1 \hat{x}' + \omega_2 \hat{y}' + \omega_3 \hat{z}',$$

has a constant value, that is

$$w = |\vec{\omega}| = \left(\omega_1^2 + \omega_2^2 + \omega_3^2\right)^{1/2} = (A^2 + \omega_3^2)^{1/2}$$

Equs. (46) and (47) are parametric equations of a circle and ω_1 and ω_2 are the components of $\vec{\omega}$ in the $x'y'$ body plane. Thus the components ω_1 and ω_2 of $\vec{\omega}$ trace out a circle in the $x'y'$ plane, which implies that the angular velocity vector $\vec{\omega}$ processes in a cone about the z' body symmetry axis with a constant angular frequency Ω as shown in Fig. 4.4. The half angle ϕ of the cone is given by

$$\tan\phi = (\omega_1^2 + \omega_2^2)^{1/2}/\omega_3 = A/\omega_3 \tag{49}$$

It should be remembered that the precession described here is relative to the body axes, which are themselves rotating in space with the larger frequency $\vec{\omega}$. From equ. (44), it is evident that the closer I_1 is to I_3, the slower would be the precession frequency Ω compared to the rotation frequency ω.

The constants A (the amplitude of precession) and ω_3 can be evaluated in terms of the rotational kinetic energy and the magnitude of the angular momentum. Thus

$$\begin{aligned} T &= \tfrac{1}{2}(\vec{w}\cdot\vec{L}) \\ &= \tfrac{1}{2}(w_{x'}\hat{x}' + w_{y'}\hat{y}' + w_{z'}\hat{z}')\cdot(L_{x'}\hat{x}' + y_{y'}\hat{y}' + L_{z'}\hat{z}') \\ &= \tfrac{1}{2}(w_{x'}\hat{x}' + w_{y'}\hat{y}' + w_{z'}\hat{z}')\cdot(I_1 w_{x'}\hat{x}' + I_2 w_{y'}\hat{y}' + I_3 w_{z'}\hat{z}') \\ &= \tfrac{1}{2}(I_1 w_{x'}^2 + I_1 w_{y'}^2 + I_1 z_{x'}^2) \\ &= \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = \tfrac{1}{2}(I_1 A^2 + I_3\omega_3^2) \end{aligned} \tag{50}$$

as $I_1 = I_2$ and

$$L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = I_1^2 A^2 + I_3^2\omega_3^2 \tag{51}$$

From the above two equations, one gets

$$A^2 = L^2 - 2I_3T/I_1(I_1 - I_3)$$

and

$$\omega_3^2 = L^2 - 2I_1T/I_3(I_3 - I_1) \tag{52}$$

The results of the above discussion could be applied in the case of the rotation of the earth about its polar axis as the external torques acting on the earth are so weak that the said motion may be considered as that of a free body or in other words earth may be considered to be a torque free rigid body. The earth is symmetrical about the polar axis and slightly flattened at the poles, resulting in an oblate spheroidal shape. This gives $I_3 > I_1$ (or I_2).

Numerically, the ratio of the moments of inertia is given by

$$(I_3 - I_1)/I_1 = 33 \times 10^{-4}$$

and hence the precession angular frequency from equ. (44) is

$$\Omega = (I_3 - I_1)/_{I_1,}\ \omega_3 \cong \omega_3/300$$

Since the period of rotation of the earth around itself ($2\pi/\omega$) is 1 day, the period of the precession of the axis of rotation about the symmetry axis is

$$\tau = 2\pi/\Omega = 2\pi/\omega_3 \times I_1/(I_3 - I_1) \cong 300 \text{ days}$$

(or about 10 months)

where we have taken$\omega_3 \cong \omega$.

The measured value is found to be ≅ 440 days. This disagreement could be because of the fact that the earth is not a perfect rigid body nor an oblate spheroid in shape. Truly speaking the shape of earth resumbles a flattened pear.

4.11 EULER'S ANGLES

The motion of a rigid body can be described by means of the three coordinates of its com and three angles specifying the orientation of the body set of axes, which are most conveniently taken as the principal axes 1, 2, 3 with origin at the com, relative to the space set of axes x, y, z. Most common angles used for the purpose are called Euler's angles.

Since we are interested only in the angles between the coordinate axes, we take the origins of two systems to coincide (point O in Fig. 4.5). The moving 1-2 plane intersects the fixed x-y plane in some line ON, called the line of nodes. This line is evidently perpendicular to both the z-axis and the 3-axis. We take the positive direction of this line as that of the vector product of $\hat{\vec{z}} \times \hat{\vec{3}}$ where $\hat{\vec{z}}$ and $\hat{\vec{3}}$ are the unit vectors along the z and 3 axes.

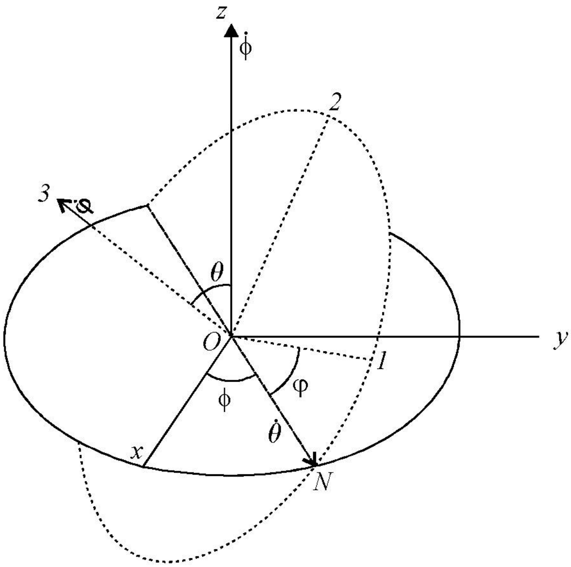


Fig. 4.5 Euler's angles and angular velocities $\dot{\phi}, \dot{\varphi}$ and $\dot{\theta}$.

In Fig. 4.5, the Euler angles θ, ϕ, and φ are shown. θ is the angle between the 3-axis and z-axis, ϕ is the angle between x-axis and ON and φ is the angle between 1-axis and ON. The angle θ takes values from 0 to π and ϕ and φ from 0 to 2 π.

Let us now express the components of the angular velocity vector $\vec{\omega}$ along the moving axes 1, 2 and 3 in terms of the Euler's angles and their derivatives. To do so, we have to find the components of the angular velocities $\dot{\theta}$, $\dot{\phi}$, $\dot{\varphi}$ along 1, 2, 3 axes. Note that the angular velocity $\dot{\theta}$ is along the line of nodes ON, the angular velocity $\dot{\phi}$ is along the z-axis, and the angular velocity $\dot{\varphi}$ is along the 3-axis. Therefore, the components of $\dot{\theta}$ along 1, 2 and 3-axes are

$$(\dot{\theta})_1 = \dot{\theta} \cos\varphi$$

$$(\dot{\theta})_2 = -\dot{\theta} \cos(90 + \varphi) = - \dot{\theta} \sin\varphi$$

$$(\dot{\theta})_3 = 0, \text{ as ON is } \perp \text{ to the 3-axis}$$

Similarly,

$$(\dot{\phi})_3 = \dot{\phi} \cos \theta,$$

and the component of $\dot{\phi}$ in the 1, 2 plane is $\dot{\phi}$ sin θ. This would be ⊥ to the line of nodes which itself is ⊥ to z as well as to 3-axis. As a result, the said component would make an angle 90-φ with the 1-axis. Thereofre

$$(\dot{\phi})_1 = (\dot{\phi} \sin \theta) \cos (90 - \varphi) = \dot{\phi} \sin \theta \sin \varphi$$

and

$$(\dot{\phi})_2 = (\dot{\phi} \sin \theta) \cos \varphi$$

Finally since the angular velocity $\dot{\varphi}$ is along the 3-axis,

$$(\dot{\varphi})_3 = \dot{\varphi}, (\dot{\varphi})_1 = (\dot{\varphi})_2 = 0$$

Collecting the components along each axis, we have

$$\omega_1 = \dot{\phi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi, \tag{53}$$

$$\omega_2 = \dot{\phi} \sin \theta \cos \varphi - \dot{\theta} \sin \varphi, \tag{54}$$

$$\omega_3 = \dot{\phi} \cos \theta + \dot{\varphi} \tag{55}$$

The above equations are called Euler's geometrical equations.

4.12 MOTION OF A HEAVY SYMMETRICAL TOP WITH LOWEST POINT FIXED

A rigid body rotating about a fixed point under the action of a torque produced by its weight (in the gravitational field) is called a heavy top. We shall limit our discussion to a special case of a symmetrical top in which $I_3 > I_1 = I_2$. The symmetry axis is of course the principal axis 3. Furthermore, the fixed point O (see Fig. 4.6) is the common origin of the moving and fixed coordinate systems and the z-axis vertical.

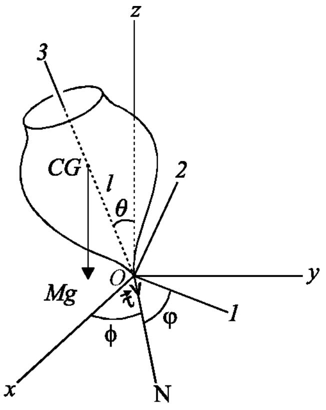


Fig. 4.6 Heavy symmetrical top with lowest point fixed.

Before we proceed further, it may be pointed out here that a wide variety of physical systems, ranging from a child's top to gyroscopic navigational instruments are approximated by such a heavy symmetrical top.

Since the point O is fixed, the configuration of the top is completely specified by the three Euler angles, θ, ϕ, and φ. The distance of the centre of gravity (CG) from the fixed point O is denoted by l. The Lagrangian method, rather than Euler's equations, would be used here to obtain solution for the motion of the top. The body being symmetrical, the kinetic energy can be written as

$$T = ½\,(I_1\,\omega_1^2 + I_2\,\omega_2^2 + I_3\,\omega_3^2)$$
$$= ½\,I_1(\omega_1^2 + \omega_2^2) + ½\,I_3\,\omega_3^2$$

Substituting ω_1, ω_2 and ω_3 from equs. (53) - (55) into above, one after simplification gets

$$T = I_1/2\,(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + I_3/2\,(\dot{\phi}\,\cos\theta + \dot{\varphi})^2 \qquad (56)$$

The potential energy in this case is given by

$$V = \text{Mg}\,l\cos\theta \qquad (57)$$

Thus the Lagrangian is

$$L = T - V = I_1/2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + I_3/2(\dot{\phi}\,\cos\theta + \quad)^2$$
$$- \text{Mg}\,l\cos\theta \qquad (58)$$

where M is the mass of the top.

Since the Lagrangian does not contain ϕ and φ, they are therefore cyclic coordinates. As a result the momenta conjugate to these coordinates are constant in time or are constants (or integrals) of motion. Thus

$$p_\phi = \partial L/\partial\dot{\phi} = (I_1\sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\varphi}\cos\theta$$
$$= \text{constant} \qquad (59)$$

$$p_\varphi = \partial L/\partial\dot{\varphi} = I_3\,(\dot{\varphi} + \dot{\phi}\,\cos\theta) = \text{constant} \tag{60}$$

There is one integral of motion: since the system is conservative, the total energy E is constant in time

$$E = T + V = I_1/2\,(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + I_3/2(\dot{\phi}\cos\theta + \dot{\varphi})^2$$
$$+ \text{M}gl\cos\theta = \text{constant} \tag{61}$$

From equ. (60), $\dot{\varphi}$ is given in terms of $\dot{\phi}$ by

$$\dot{\varphi} = p_\varphi/I_3 - \dot{\phi}\cos\theta \tag{62}$$

The above result can be substituted in equ. (59) to eliminate $\dot{\varphi}$. Thus

$$p_\phi = \text{I}_1\dot{\phi}\sin^2\theta + \text{I}_3\,\dot{\phi}\cos^2\theta + I_3\cos\theta\,(p_\varphi/I_3 - \dot{\phi}\,\cos\theta)$$
$$= I_1\dot{\phi}\sin^2\theta + p_\varphi\cos\theta$$

or
$$\dot{\phi} = \frac{p_\phi - p_\varphi\cos\theta}{I_1\sin^2\theta} \tag{63}$$

which furnishes the dependence of ϕ on time if θ were known as a function of time.

Substituting above equation back into equ. (62) gives

$$\dot{\varphi} = p_\varphi/I_3 - \frac{p_\phi - p_\varphi\cos\theta}{I_1\sin^2\theta}\cos\theta \tag{64}$$

which furnishes φ as a function of time if θ is known as a function of time.

Finally we use equs. (63) and (60) to eliminate $\dot{\phi}$ and $\dot{\varphi}$ from the energy equation, resulting in the following differential equation involving θ alone.

$$E = \tfrac{1}{2}I_1\dot{\theta}^2 + \frac{(p_\phi - p_\varphi\;\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\varphi^2}{2I_3} + \text{M}g\,l\cos\theta \tag{65}$$

Since $p_\varphi(= I_3\,\omega_3$, ω_3 being constant in time) is constant in time, therefore $E - p_\varphi^2/2I_3$ is a constant of motion, which we shall denote by E'. As a result, the energy equation can be written as

$$E' = \tfrac{1}{2}I_1\dot{\theta}^2 + \frac{(p_\phi - p_\varphi\cos\theta)^2}{2I_1\sin^2\theta} + \text{M}g\,l\cos\theta \tag{66}$$

The above equation has the form of an equivalent one-dimensional problem in the variable θ with the effective potential energy $V_{eff}(\theta)$ (or V') given by

$$V_{eff}(\theta) = \text{M}g\,l\cos\theta + \frac{(p_\phi - p_\varphi\;\cos\theta)^2}{2I_1\sin^2\theta} \tag{67}$$

We would now proceed to use this one-dimensional problem to discuss the motion of θ. Since V_{eff} assumes infinite values for $\theta = 0$ and π, and finite values for all other values of θ, it implies that for some value of θ between 0 and π, V_{eff} has a minimum value (see Fig. 4.7). This occurs at the value of θ_0 which is root of the equation.

$dV_{eff}/d\theta = 0$

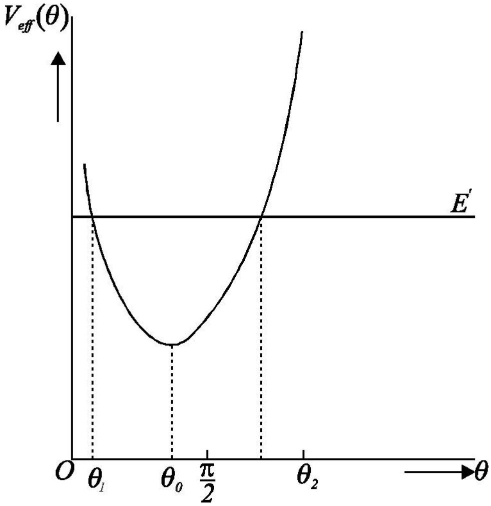


Fig. 4.7 $V_{\text{eff}}(\theta)$ versus θ $(0 \leq \theta \leq \pi)$ for a heavy symmetrical top.

$$= -\text{Mg}\, l \sin\theta$$

$$+ \frac{1}{2I_1} \frac{\sin^2\theta \times 2 \times (p_\phi - p_\varphi \cos\theta) p_\varphi \sin\theta - (p_\phi - p_\varphi \cos\theta)^2 2\sin\theta\cos\theta}{\sin^4\theta}$$

or

$$\sin\theta \left[\frac{2\sin^2\theta p_\varphi (p_\phi - p_\varphi \cos\theta) - (p_\phi - p_\varphi \cos\theta)^2 2\cos\theta}{2I_1 \sin^4\theta} - Mg\text{l} \right] = 0 \quad (68)$$

As a resuilt,

$$\sin\theta = 0 \quad (69)$$

(which corresponds to $\theta = 0$ and π) gives the maximum. Hence

$$\frac{\sin^2\theta_0\; p_\varphi (p_\phi - p_\varphi \cos\theta_0) - (p_\phi - p_\varphi \cos\theta_0)^2 \cos\theta_0}{I_1 \sin^4\theta_0} - Mgl = 0 \quad (70)$$

must correspond to minimum.

In general, therefore, the θ - motion is confined (or bound) between the two values θ_1 and θ_2 (called turning points), which are roots of the equation.

$$E' = \text{V}_{\text{eff}}(\theta)$$

or

$$E' = \text{Mg}\, l \cos\theta + \frac{(p_\phi - p_\varphi \cos\theta)^2}{2I_1 \sin^2\theta} \quad (71)$$

Note that the above equation being cubic in $\cos\theta$, would have three roots. However, the V_{eff} curve (Fig. 4.7) indicates that only two out of three roots would lie in the physically possible range of $\cos\theta$, that is, –1 and +1 and the third must be trivial.

Thus the symmetry axis of the top oscillates between two limiting angles θ_1 and θ_2 for a given value of energy E'. This motion of the top (or variation in angle θ) is referred to as the mutation of the symmetry axis of the top.

If $$E' = \mathrm{V}_{\mathrm{eff}}(\theta_0) \tag{72}$$

then θ remains fixed at θ_0 (or the axis of symmetry is inclined at angle θ_0), and the top (or axis of the top) precesses about the space z-axis (or vertical) with the constant angular velocity ($\dot{\phi} = \dot{\phi}_0$ corresponding to $\theta = \theta_0$)

$$\dot{\phi}_0 = \frac{p_\phi - p_\varphi \cos\theta_0}{I_1 \sin^2\theta_0} \tag{73}$$

Further, we find that for a given value of θ_0 there exist two values of $\dot{\phi}_0$ which would satisfy equ. (70) which can be written as

$$p_\varphi \frac{p_\phi - p_\varphi \cos\theta}{I_1 \sin^2\theta} - \cos\theta_0\, I_1 \left(\frac{p_\phi - p_\varphi \cos\theta_0}{I_1 \sin^2\theta_0}\right)^2 - \mathrm{Mg}l = 0 \tag{74}$$

Putting the value of $\dot{\phi}_0$ from equ. (73) into equ. (74) one gets

$$p_\varphi \dot{\phi}_0 - \cos\theta_0\, I_1 \dot{\phi}_0^2 - \mathrm{Mg}l = 0$$

or $$I_3(\dot{\phi}_0 \cos\theta_0 + \dot{\varphi})\, \dot{\phi}_0 - \cos\theta_0\, I_1\, \dot{\phi}_0^2 - \mathrm{Mg}l = 0$$

where we have substituted for p_φ from equ. (60). As a result

$$\dot{\phi}_0^2(I_1 - I_3)\cos\theta_0 - \dot{\phi}_0 \dot{\varphi}\, I_3 + \mathrm{Mg}l = 0 \tag{75}$$

The two values of $\dot{\phi}_0$ which satisfy the above equation, are

$$\dot{\phi}_0 = \frac{I_3\dot{\varphi} \pm \sqrt{I_3^2\dot{\varphi}^2 - 4Mgl(I_1 - I_3)\cos\theta_0}}{2(I_1 - I_3)\cos\theta_0} \tag{76}$$

These two values of $\dot{\phi}_0$, one corresponding to the (+) sign and the other corresponding to the (–) sign for which the symmetry axis of the top would precess uniformly at the constant angle θ_0 are referred to as the fast and slow precessional angular velocities, respectively.

Since $\dot{\phi}_0$ must be real, it is, therefore, evident that the quantity under the square root sign in equ. (76) must be positive definite, that is

$$I_3^2\dot{\varphi}^2 - 4\,\mathrm{Mg}\,l\,(I_1 - I_3)\cos\theta_0 \geq 0$$

or $$\dot{\varphi}^2 \geq \frac{4Mgl(I_1 - I_3)\cos\theta_0}{I_3^2} \tag{77}$$

The above equation limits the values of $\dot{\varphi}$, called the spin angular velocity about the symmetry (or 3) axis for which steady precession may occur at the angle θ_0.

As discussed earlier in connection with the $V_{eff}(\theta)$ versus θ (equ. 67) plot (Fig. 4.7) which had minimum at $\theta = \theta_0$ corresponding to the point of stable equilibrium, the motion of the axis of symmetry is limited between the turning points θ_1 and θ_2 corresponding to the energy E' of the top. It, therefore, follows that situations exist for which the mutational motion of the axis of symmetry of the top can be described as a small simple harmonic oscillation about the value $\theta = \theta_0$. The frequency of these small oscillations would be given by (see equ. 2, sec. 5.1 in the case of oscillating mass)

$$\omega = \left[\frac{1}{I_1}\frac{d^2V_{eff}(\theta)}{d\theta^2}\right]^{1/2}_{\theta=\theta_0} \tag{78}$$

which can be found out by calculating $d^2 V_{eff}(\theta)/d\theta^2\big|_{\theta=\theta_0}$ using equ. (67). The result contains $\dot{\phi}_0$ (and $\dot{\phi}_0^2$) which can be substituted from equ. (73). The final outcome on substituting in equ. (78) gives ω.

Further, we note, according to equ. (63), that the angular velocity of the symmetry axis about the vertical (3-axis), $\dot{\phi}$, is zero when $\cos\theta = p_\phi/p_\varphi$. If θ lies outside the range between θ_1 and θ_2, then $\dot{\phi}$ never vanishes, and the symmetry axis of the top precesses round the vertical in a fixed direction, and at the same time wobbles (oscillates) up and down between θ_1 and θ_2 (the said oscillation is called mutation). This motion is illustrated in Fig. 4.8(*a*), which shows the path described by the symmetry axis of the top when projected on a sphere of unit radius about the fixed point.

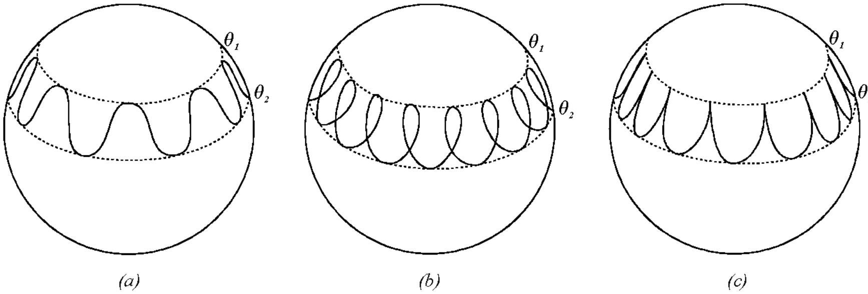


Fig. 4.8 θ motion (mutation). The diagrams show the path of the symmetry axis of the top as projected on a sphere of unit radius about the fixed point (a) precession round the vertical in a fixed direction, and wobbling up and down between θ_1 and θ_2 (b) looping motion, and (c) cusplike motion.

On the other hand, if $\theta_1 < \cos^{-1}(p_\phi/p_\varphi) < \theta_2$, the symmetry axis moves in loops, as shown in Fig. 4.8(b). The angular velocity $\dot{\phi}$ has one sign near the top of the loop, and opposite sign near the bottom.

The limiting case between the two kinds of motion occurs when $\cos^{-1}(p_\phi/p_\varphi) = \theta_1$, that is $(p_\phi = p_\varphi \cos\theta)|_{\theta=\theta_1} = 0$. From equs. (63) and (64), this amounts to $\dot{\phi}|_{\theta=\theta_1} = 0$,

$\dot{\varphi} = \omega_3$ and $\dot{\theta}|_{\theta=\theta_1} = 0$ as θ is fixed. As a result, the loops shrink to cusp as shown in Fig. 4.8(c). The symmetry axis of the top comes instantaneously to rest at the top of each loop. This kind of motion would occur if the top is set spinning with its symmetry axis initially at rest. The above mentioned kinds of motion can be observed using a small gyroscope. However, the type of motion would change slowly with time because of the presence of frictional forces.

PROBLEMS

4.1 Find the principal moments of inertia about the com of a flat rigid body in the shape of 45° right triangle with uniform mass density. What are the principal axes?

4.2 A uniform rod slides with its ends on a smooth vertical circle. If the rod subtends an angle of 120° at the centre of the circle, show that the equivalent simple pendulum has a length equal to the radius of the circle.

4.3 Three equal mass points having each one mass m are located at $(a, 0, 0)$, $(0, a, 2a)$ and $(0, 2a, a)$. Find the principal moments of inertia about the origin and a set of principal axes.

4.4 Obtain the principal moments of inertia about com of a flat rigid body in the shape of an isosceles right - angled triangle with uniform mass density.

4.5 Using Euler's equations of motion, obtain condition for the uniform precession of a symmetrical top in a gravitational field by imposing the requirement that the motion be a uniform precession without mutation.

4.6 Show that the magnitude of the angular momentum for a heavy symmetrical top can be expressed as a function of θ and the constants of the motion.

4.7 A uniform bar of mass M and length $2l$ is suspended from one end by a spring of force constant k. The bar can swing freely only in one vertical plane, and the spring is constrained to move only in the vertical direction (see Fig. 4.12). Set up the equations of motion in the Lagrangian formulation.

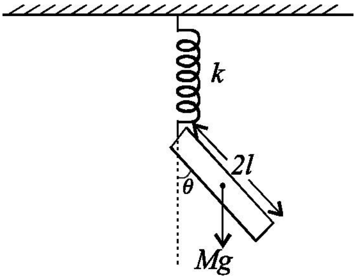


Fig. 4.12 A bar of mass M and length 2 *l* swinging in a vertical plane and the spring constrained to move in a vertical direction.

4.8 A car starts from rest with one of its doors initially at right angle. The door slams shut as the car picks up speed. Obtain an expression for the time needed for the door to close if the acceleration f is constant, the radius of gyration of the door about the axis of rotation is r_0, and the centre of mass is at a distance a from the hinges.

4.9 Consider a homogeneous cube of density ρ, mass M and side L. The origin of the coordinate system is located at one corner of the cube, and the three adjacent edges lie along the coordinate axes as shown in Fig. 4.14. From the arrangement, show that the inertia tensor is given by

$$I = ML^2 \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}$$

There from find the principal moments of inertia.

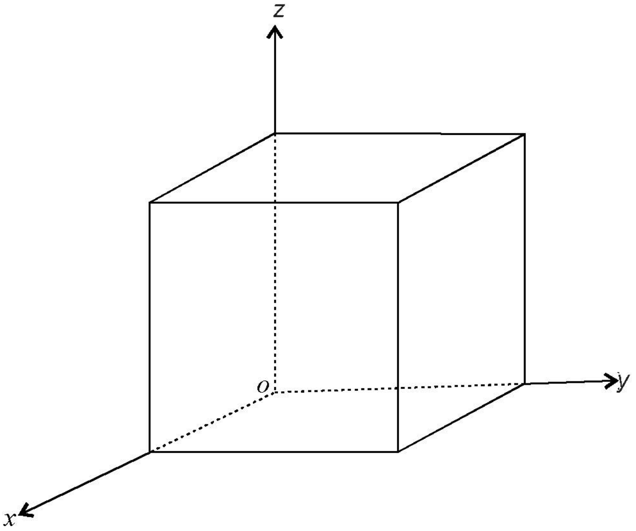


Fig. 4.14 Problem 4.9

4.10 Consider a homogenous cube of density ρ, mass M and side L. For a coordinate system with origin at com (instead of at O, problem 4.9) of the cube, find the elements of inertia tensor.

SOLUTIONS

4.1 Let ρ be the uniform mass density of the given flat rigid body (see Fig. 4.9) with com at O ($DO = l/3$, $OA = 2l/3$, where $l = BD = DA$). dxdy represents the mass element around x, y. Therefore, the moment of inertia about the x-axis.

$$I_{xx} = \int\int \rho \, dx \, dy \, y^2$$

$$= \rho \int dx \int y^2 \, dy \tag{79}$$

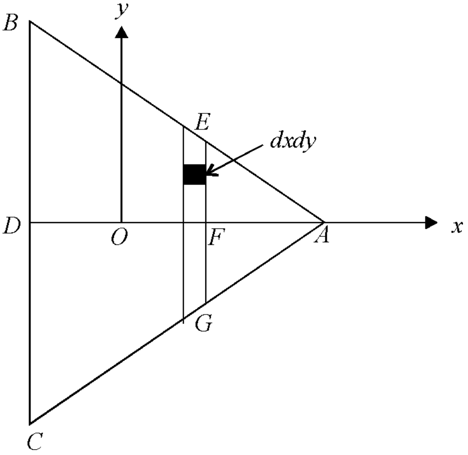


Fig. 4.9 A flat rigid body in the shape of a 45° right triangle. O is the com of the body.

From the triangles ADB and AFE,

$$BD/EF = DA/FA$$

or

$$l/EF = l/(2l/3 - x)$$

Therefore,

$$EF = y = 2l/3 - x$$

As a result, the limits for y in equ. (79) can be taken as- $(2l/3 - x)$ to $+ (2l/3 - x)$ and for x from $-l/3$ to $+2l/3$. Hence

$$I_{xx} = \rho \int_{-l/3}^{+2l/3} dx \int_{-(2l/3-x)}^{+(2l/3-x)} y^2 dx$$

$$= \rho \int_{-l/3}^{+2l/3} dx \left[y^3/3 \right]_{-(2l/3-x)}^{+(2l/3-x)} dx$$

$$= 2\rho/3 \int_{-l/3}^{+2l/3} (2l/3 - x)^3 \, dx$$

$$= 2\rho/3 \times -1/4 \left[(2l/3 - x)^4 \right]_{-l/3}^{+2l/3}$$

$$= 1/6 \, \rho l^4 \tag{80}$$

The moment of inertia about the y-axis

$$I_{yy} = \rho \int\int dx \, dy \, x^2$$

$$= \rho \int_{-l/3}^{+2l/3} x^2\,dx\; y\Big|_{-(2l/3-x)}^{+(2l/3-x)}$$

$$= 2\rho \int_{-l/3}^{+2l/3} x^2(2l/3 - x)dx$$

$$= \rho l^4/18 \tag{81}$$

Further using the perpendicular axis theorem, that is, the sum of the moments of inertia of a plane lamina about any two perpendicular axes in the plane of lamina is equal to the moment of inertia about an axis passing through the point of intersection and perpendicular to the plane of lamina. Therefore, the moment of inertia about the z-axis,

$$I_{zz} = I_{xx} + I_{yy} = 2/9\,\rho\, l^4 \tag{82}$$

The products of inertia, in this case, are given by

$$I_{xy} = -\rho \int\!\!\int dx\,dy\,xy = -\rho \int x\,dx\; y^2\Big|_{-(2l/3-x)}^{+(2l/3-x)} = 0$$

and similarly

$$I_{yz} = I_{xz} = 0$$

or

$$I_{xy} = I_{yz} = I_{xz} = 0 \tag{83}$$

which shows that all the off-diagonal elements are zero implying that the set of axes chosen are nothing but the principal axes and hence I_{xx}, I_{yy} and I_{zz} are the principal moments of inertia.

4.2 Equation of motion of the given rod (Fig. 4.10) is given by

$$dL/dt = \frac{dI\dot{\theta}}{dt} = N^e$$

$$= -\mathrm{mg}\, h \sin\theta$$

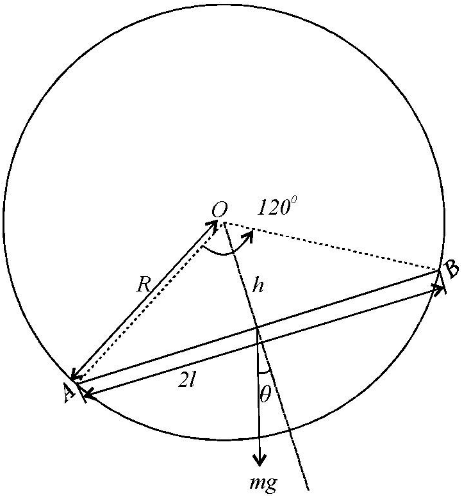


Fig. 4.10 A uniform rod sliding on a smooth vertical circle.

where L and N^e represent the total angular momentum and the external torque about the com, respectively. θ is the angle which a line normal to the rod makes with the vertical. I being the moment of inertia of the rod about an axis passing through the centre of the circle. The negative sign signifies the fact that torque acts in such a way as to decrease the angle θ.

From above,

$$I\ddot{\theta} + mg\,h\sin\theta = 0$$

or $$I\ddot{\theta} + mg\,R\cos 60^\circ \sin\theta = 0 \tag{84}$$

Now using the parallel axis theorem, that is, the moment of inertia of a body about a given axis is equal to the moment of inertia about a parallel axis through the com plus the moment of inertia of the body, as if whole mass of the body is located at the com, with respect to the original axis, one gets

$$I = 1/3\ m\,l^2 + m\,h^2,$$

where $2l$ is the length of rod.

or $$I = 1/3\ m(R\cos 30^\circ)^2 + m(R\cos 60^\circ)^2$$

$$= 1/2\ m\,R^2 \tag{85}$$

Substituting the value of I from above into equ. (84), we get

$$1/2\ m\,R^2\ddot{\theta} + mg\,R/2\sin\theta = 0$$

or $$\ddot{\theta} + g/R\sin\theta = 0 \tag{86}$$

Assuming θ to be small, $\sin\theta \cong \theta$, the above equation takes the form

$$\ddot{\theta} + g/R\,\theta = 0 \tag{87}$$

which is the equation of a simple pendulum having length equal to the radius of the circle.

4.3 The moments of inertia about x, y and z axes are given by

$$\begin{aligned} I_{xx} &= \sum_{i=1}^{3} m\,(y_i^2 + z_i^2) \\ &= m[(0+0) + (a^2 + 4a^2) + (4a^2 + a^2)] \\ &= 10\,m\,a^2, \end{aligned}$$

$$\begin{aligned} I_{yy} &= \sum_{i=1}^{3} m\,(z_i^2 + x_i^2) \\ &= m[(0 + a^2) + (4a^2 + 0) + (a^2 + 0)] \\ &= 6\,m\,a^2, \end{aligned}$$

$$\begin{aligned} I_{zz} &= \sum_{i=1}^{3} m\,(x_i^2 + y_i^2) \\ &= m[(a^2 + 0) + (0 + a^2) + (0 + 4a^2)] \\ &= 6\,m\,a^2 \end{aligned}$$

The products of inertia are given by

$$I_{xy} = -\sum_{i=1}^{3} m x_i y_i = -m\,(0 + 0 + 0) = 0,$$

$$I_{yz} = -\sum_{i=1}^{3} m\, y_i z_i = -m\,(0 + 2a^2 + 2a^2) = -4\, ma^2,$$

$$I_{zx} = -\sum_{i=1}^{3} m\, z_i x_i = -m\,(0 + 0 + 0) = 0$$

In order to find the principal moments of inertia, we write the characteristic (or secular) equation,

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - I & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - I \end{vmatrix} = 0$$

Substituting the values of I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, and I_{zx} (or I_{xz}), we get

$$\begin{vmatrix} 10ma^2 - I & 0 & 0 \\ 0 & 6ma^2 - I & -4ma^2 \\ 0 & -4ma^2 & 6ma^2 - 1 \end{vmatrix} = 0 \qquad (88)$$

which gives

$$(10\, ma^2 - I)\,[(6ma^2 - I)^2 - (-4ma^2)^2] = 0,$$

therefore $I = I_1 = 10\, ma^2,$

and $6ma^2 - I = \pm\, 4ma^2$

or $I = I_2 = 2\, ma^2,\, I = I_3 = 10\, ma^2$

Hence the principal moments of inertia are $10\, ma^2$, $2\, ma^2$ and $10\, ma^2$.

Let us now proceed for the determination of principal axes. Since $\boldsymbol{I}$ is a tensor

$$\boldsymbol{I}\, a = I\, a,$$

where I is called the eigen value and a the eigen vector of $\boldsymbol{I}$. The equation is called the eigen value equation which can be written as

$$(\boldsymbol{I} - I\,\mathbf{1})a = 0 \qquad (89)$$

The above equation corresponds to a set of the three homogeneous equations for the components of the eigen vector. Note that **1** here corresponds to the unit matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and 0 to the null matrix (column). Equ. (88) is nothing but the secular equation, in the form of a cubic in I, whose three roots are the desired principal moments of inertia. For each of these roots the equation

$$I a_j = I_j a_j \tag{90}$$

can be solved to find the direction of the corresponding principal axis (a_1, a_2 and a_3 are the components of the eigen vector a)

The eigen vector (or a_1 or principal axis) corresponding to the eigen value I_1 = 10 ma^2 can be found from equ. (89), that is

$$\begin{bmatrix} 10ma^2 - 10ma^2 & 0 & 0 \\ 0 & 6ma^2 - 10ma^2 & -4ma^2 \\ 0 & -4ma^2 & 6ma^2 - 10ma^2 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which simply amounts to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the normalized eigen vector corresponding to I_1 = 10 ma^2 is

$$\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \tag{91a}$$

Similarly corresponding to I_2 = 2 ma^2, one gets

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies that the corresponding normalized eigen vector is

$$\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \tag{91b}$$

Further corresponding to I_3 = 10 ma^2,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies that the corresponding normalized eigen vector is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{91c}$$

4.4 Let ρ be the uniform mass density of the given flat rigid body (see Fig. 4.11) with com at $O'(a/3, a/3)$. $OA = OB = a$, $dxdy$ represents the mass element around

x,y. Therefore, the moment of inertia of the body about x-axis with origin at $O(0, 0)$,

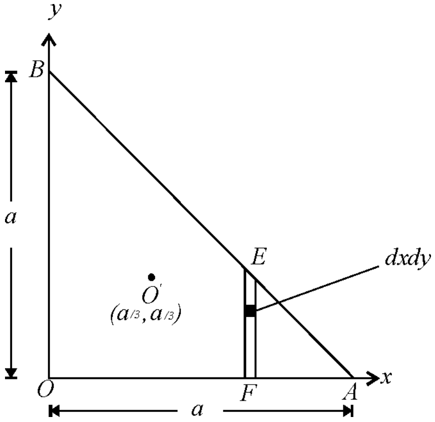


Fig. 4.11 A flat rigid body in the shape of an isosceles right-angled triangle. O is the origin and O' the com of the body.

$$I_{xx}(O) = \int\int \rho \, dx \, dy \, y^2$$
$$= \rho \int dx \int y^2 \, dy \qquad (92)$$

From the triangles AOB and AFE,

$$OB/EF = OA/FA$$

or $$a/EF = a/(a-x)$$

Therefore,

$$EF = y = (a-x)$$

As a result, the limits for y in equ. (92) can be taken as 0 to $a - x$ and for x from O to a. Hence.

$$I_{xx}(O) = \rho \int_0^a dx \int_0^{a-x} y^2 dy$$
$$= \rho/3 \int_0^a (a-x)^3 dx$$
$$= \rho/3 \, (a-x)^4/4 \Big|_0^a$$
$$= \rho a^4/12 \qquad (93a)$$

The moment of inertia about y-axis with origin at O(0, 0),

$$I_{yy}(O) = \rho \int\int dx \, dy \, x^2 = \rho \int_0^a x^2 \, dx \int_0^{a-x} dy$$
$$= \rho \int_0^a x^2 \, (a-x) \, dx$$

$$= \rho a^4/12 \tag{93b}$$

Now using the perpendicular axis theorem, the moment of inertia about the z-axis with origin at $O(0, 0)$,

$$I_{zz}(O) = I_{xx}(O) + I_{yy}(O) = \rho a^4/6 \tag{93c}$$

Using the parallel axis theorem, the moments of inertia about com of the body would be given by

$$\rho a^4/12 = \mathrm{I}_{xx}\,(\text{com}) + \rho a^2/2 \times (a/3)^2$$

or
$$I_{xx}\,(\text{com}) = \rho a^4/12 - \rho a^2/2 \times (a/3)^2 = \rho a^4/36, \tag{94a}$$

$$I_{yy}\,(\text{com}) = \rho a^4/12 - \rho a^2/2 \times (a/3)^2 = \rho a^4/36, \tag{94b}$$

Further using the perpendicular axis theorem,

$$\begin{aligned} I_{zz}\,(\text{com}) &= I_{xx}\,(\text{com}) + I_{yy}\,(\text{com}) \\ &= \rho a^4/36 + \rho a^4/36 = \rho a^4/18 \end{aligned} \tag{94c}$$

The product of inertia w.r.t. O (0, 0),

$$\begin{aligned} I_{xy} &= -\int \rho\, dxdy\, xy \\ &= -\rho \int_0^a x\, dx \int_0^{a-x} y\, dy \\ &= -\rho/2 \int_0^a x(a-x)^2\, \mathrm{d}x \\ &= -\rho a^4/24 \end{aligned} \tag{95}$$

Using once again the parallel axis theorem,

$$I_{xy}(O) = -MXY + I_{xy}\,(\text{com})$$

where M is the total mass of the body and X, Y the coordinates of the com.

Therefore,

$$\begin{aligned} I_{xy}(\text{com}) &= I_{xy}(O) + MXY \\ &= -\rho a^4/24 + \rho a^2/2 \times a/3 \times a/3 \\ &= \rho a^4/72 \end{aligned} \tag{96a}$$

Further,

$$I_{xz}(\text{com}) = I_{yz}(\text{com}) = 0 \tag{96b}$$

The inertia tensor I, in the present case, therefore reads

$$\boldsymbol{I} = \rho a^4/72 \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{97}$$

To find the eigen values of $\boldsymbol{I}$ (or principal moments of inertia), we write the secular equation,

$$\begin{vmatrix} 2-I & 1 & 0 \\ 1 & 2-I & 0 \\ 0 & 0 & 4-I \end{vmatrix} = 0 \tag{98}$$

where the three roots, $I = I_1$, I_2, and I_3 would give the values of principal moments of inertia. From above

$$(2 - I)[(2 - I(4-I)] - (4 - I) = 0$$

or $\quad (4 - I)\,[(2 - I)^2 - 1] = 0$

or $\quad I_1 = 4 \times \rho a^4/72,$

$$I_2 = 1 \times \rho a^4/72, \quad I_3 = 3 \times \rho a^4/72 \tag{99}$$

The principal axis corresponding to $I_1 = 4 \times \rho a^4/72$ can be found from the equation

$$\begin{bmatrix} 2-4 & 1 & 0 \\ 1 & 2-4 & 0 \\ 0 & 0 & 4-4 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which simply amounts to

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the normalized eigen vector corresponding to $I_1 = 4 \times \rho a^4/72$ is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Similarly, corresponding to $I_2 = 1 \times \rho a^4/72$, one gets

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies that the corresponding normalized eigen vector is

$$\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

Further corresponding to $I_3 = 3 \times \rho a^4/72$,

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which implies that the corresponding normalized eigen vector is

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

4.5 The Euler's equations of motion are given by

$$N_1 = I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3), \tag{100}$$

$$N_2 = I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1), \tag{101}$$

$$N_3 = I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) \tag{102}$$

in which N_1, N_2, and N_3 are components of the torque $\vec{\tau}$

$$\vec{\tau} = \vec{r} \times M\vec{g} \tag{103}$$

(which is along the line of nodes) along the body axes (see Fig. 4.6) and are given by

$$N_1 = Mgl \sin\theta \cos\varphi, \tag{104}$$

$$N_2 = \mathrm{Mg}l \sin\theta \sin\varphi, \tag{105}$$

$$N_3 = 0 \tag{106}$$

Further, ω_1, ω_2, and ω_3 are given by

$$\omega_1 = \dot{\phi} \sin\theta \sin\varphi + \dot{\theta} \cos\varphi, \tag{53}$$

$$\omega_2 = \dot{\phi} \sin\theta \cos\varphi - \dot{\theta} \sin\varphi, \tag{54}$$

$$\omega_3 = \dot{\phi} \cos\theta + \dot{\varphi} \tag{55}$$

Also use the facts that the top is symmetrical, that is, $I_1 = I_2$ and that the motion is uniform precession without mutation, which results in

$$\dot{\phi} = \text{constant}, \tag{107}$$

and $\dot{\theta} = 0$ (or θ is a constant $= \theta_0$) (108)

Using equs. (104), (105) and conditions (107) and 108), equs. (100) and (101) result in

$$I_1(\dot{\phi} \sin\theta \cos\varphi \dot{\varphi}) - \dot{\phi} \sin\theta \cos\varphi\, (\dot{\phi} \cos\theta + \dot{\varphi})\, (I_1 - I_3) = \mathrm{Mg}\, l \sin\theta \cos\varphi$$

or $I_1 \dot{\phi} \dot{\varphi} \cos\varphi - \dot{\phi} \cos\varphi\, (\dot{\phi} \cos\theta + \dot{\varphi})\, (I_1 - I_3) = \mathrm{Mg}l \cos\varphi$ (109)

and

$$I_1\, (-\dot{\phi} \sin\theta \sin\varphi\, \dot{\varphi}) + (\dot{\phi} \cos\theta + \dot{\varphi})\, \dot{\phi} \sin\theta \sin\varphi\, (I_1 - I_3) = \mathrm{Mg}\, l \sin\theta \sin\varphi$$

or

$$I_1\, (-\dot{\phi} \dot{\varphi} \sin\varphi) + \dot{\phi} \sin\varphi\, (\dot{\phi} \cos\theta + \dot{\varphi})\, (I_1 - I_3) = \mathrm{Mg}l \sin\varphi \tag{110}$$

Squaring equs. (109) and (110) and then adding, one gets

$$I_1^2\dot{\phi}^2\dot{\varphi}^2+\dot{\phi}^2(\dot{\phi}\cos\theta+\dot{\varphi})^2\,(I_1-I_3)^2-2I_1\,\dot{\phi}\dot{\varphi}\,\dot{\phi}(\dot{\phi}\,\cos\theta+\dot{\varphi})\,(I_1-I_3)=M^2g^2l^2$$

or $\quad I_1\dot{\phi}\dot{\varphi}-\dot{\phi}(\dot{\phi}\cos\theta+\dot{\varphi})\,(I_1-I_3)=Mgl$

or $\quad \dot{\phi}\,[I_3\dot{\varphi}-(I_1-I_3)\,\dot{\phi}\cos\theta]=Mgl \qquad (111)$

which is same as equ. (75).

4.6 The total angular momentum is given by

$$L^2=I_1^2\,\omega_1^2+I_2^2\,\omega_2^2+I_3^2\,\omega_3^2 \tag{112}$$

where I_1, I_2 and I_3 are the principal moments of inertia and ω_1, ω_2, ω_3 are the components of the angular velocity along the principal axes.

In the present case of a symmetrical top, $I_1=I_2$, therefore

$$L^2=I_1^2\,(\omega_1^2+\omega_2^2)+I_3^2\,\omega_3^2 \tag{113}$$

After substituting the values for ω_1, ω_2, and ω_3 from equs. (53) - (55), one gets

$$\begin{aligned}
L^2 &= I_1^2\,(\dot{\phi}^2\sin^2\theta+\dot{\theta}^2)+I_3^2(\dot{\phi}^2\cos^2\theta+\dot{\varphi}^2+2\dot{\phi}\dot{\varphi}\,\cos\theta)\\
&=\dot{\phi}^2(I_1^2\sin^2\theta+I_3^2\cos^2\theta)+I_1^2\dot{\theta}^2+I_3^2(\dot{\varphi}^2+2\dot{\phi}\dot{\varphi}\,\cos\theta)\\
&=\left(\frac{b-a\cos\theta}{\sin^2\theta}\right)^2(I_1^2\sin^2\theta+I_3^2\cos^2\theta)+I_1^2\dot{\theta}^2+I_3^2[(I_1a/I_3-\cos\theta\,\frac{b-a\cos\theta}{\sin^2\theta})^2\\
&\quad+2\,\frac{b-a\cos\theta}{\sin^2\theta}\times(I_1a/I_3-\cos\theta\,\frac{b-a\cos\theta}{\sin^2\theta})\,\cos\theta]
\end{aligned} \tag{114}$$

where the two constants of motion p_φ and p_ϕ are expressed in terms of new constants a and b such that

$$p_\varphi=I_1a,\ p_\phi=I_1b \tag{115}$$

and values of $\dot{\phi}$ and $\dot{\varphi}$ are substituted from equs. (63) and (64).

Further putting the value of $\dot{\theta}^2$ from equ. (65) to (114) after making the substitutions,

$$\alpha=2E'/I_1,\quad \beta=2\ \mathrm{Mg}l/I_1 \tag{116}$$

where α and β are two new constants, one gets

$$\begin{aligned}
L^2 &=\left(\frac{b-a\cos\theta}{\sin^2\theta}\right)^2(I_1^2\sin^2\theta+I_3^2\cos^2\theta)+I_1^2\left[(\alpha-\beta\cos\theta)-\left(\frac{b-a\cos\theta}{\sin\theta}\right)^2\right]\\
&\quad+I_3^2\left[\left(\frac{I_1a}{I_3}-\cos\theta\frac{b-a\cos\theta}{\sin^2\theta}\right)^2+2\frac{b-a\cos\theta}{\sin^2\theta}\times\left(\frac{I_1a}{I_3}-\cos\theta\frac{b-a\cos\theta}{\sin^2\theta}\right)\cos\theta\right]\\
&=\left(\frac{b-a\cos\theta}{\sin^2\theta}\right)^2(I_1^2\sin^2\theta+I_3^2\cos^2\theta-I_1^2\sin^2\theta-2\,I_3^2\cos^2\theta)\\
&\quad+I_1^2\,(\alpha-\beta\cos\theta)+I_1^2a^2
\end{aligned}$$

Therefore, the angular momentum is given by

$$L = [I_1^2(\alpha - \beta \cos\theta) + I_1^2 a^2 - \left(\frac{b - a\cos\theta}{\sin^2\theta}\right)^2 I_3^2 \cos^2\theta]^{1/2} \quad (117)$$

Further note that if θ is constant,

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} = \text{constant} \quad (118)$$

4.7 Let $\bar{x}$, $\bar{y}$ denote the coordinates of the com of the bar and x the extension in the string from its equilibrium (unstretched) value. Therefore, the kinetic energy of translation of the bar would be given by

$$\begin{aligned} T_{\text{trans}} &= \tfrac{1}{2} M(\dot{\bar{x}}^2 + \dot{\bar{y}}^2) \\ &= \tfrac{1}{2} M[(\dot{x} - l\dot{\theta}\sin\theta)^2 + (l\dot{\theta}\cos\theta)^2] \\ &= \tfrac{1}{2} M(\dot{x}^2 + l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\sin\theta) \end{aligned}$$

where

$\bar{x} = x + l\cos\theta$, $\bar{y} = l\sin\theta$, $2l$ being the length of bar.

The kinetic energy due to the rotation of the bar would be given by

$$T_{rot} = 1/2\, I\omega^2 = 1/2 \times 1/3\, Ml^2 \times \dot{\theta}^2 = 1/6\, M l^2 \dot{\theta}^2$$

where I is the moment of inertia of the bar about its axis of rotation.

The potential energy of the system is given by

$$V = -Mg(x + l\cos\theta) + \tfrac{1}{2} kx^2$$

As a result, the Lagrangian of the system would be given by

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2} M(\dot{x}^2 + 4/3\, l^2\dot{\theta}^2 - 2l\dot{x}\dot{\theta}\sin\theta) + Mg(x + l\cos\theta) - \tfrac{1}{2} kx^2 \end{aligned}$$

where $T = T_{\text{trans}} + T_{\text{rot}}$

Hence the equations of motion are

$$\ddot{x} - l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta = g - k/M\ x \quad (119)$$

and

$$d/dt(4/3\, M l^2\dot{\theta} - Ml\dot{x}\sin\theta) = -Mg\, l\sin\theta - Ml\dot{x}\dot{\theta}\cos\theta$$

or $4/3\, l\ddot{\theta} - \ddot{x}\sin\theta = -g\sin\theta \quad (120)$

Multiplying equ. (119) by $\sin\theta$ and then adding to equ. (120), one gets

$$-l\ddot{\theta}\sin^2\theta - l\dot{\theta}^2\sin\theta\cos\theta + 4/3\, l\ddot{\theta} = g\sin\theta - k/M\ x\sin\theta - g\sin\theta$$

or $l\ddot{\theta}(4/3 - \sin^2\theta) - l\dot{\theta}^2\sin\theta\cos\theta = -k/M\sin\theta \quad (121)$

4.8 The moment of inertia of the door about the axis of rotation, from the parallel axis theorem is given by (See Fig. 4.13)

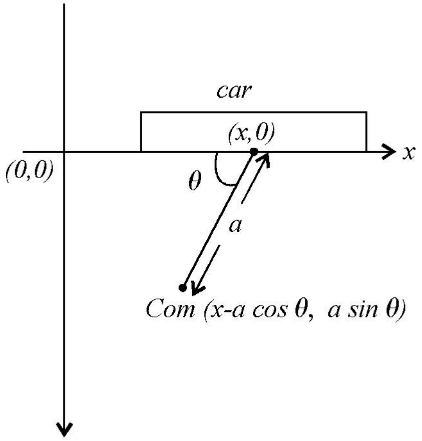


Fig. 4.13 Problem 4.8

$$mr_0^2 = I_{CM} + ma^2$$

or $I_{CM} = m(r_0^2 - a^2)$

where r_0 is the radius of gyration of the door about the axis of rotation.

The kinetic energy of the door

$$T = T_{\text{trans}} + T_{\text{rot}}$$

$$= ½\, m\{d/dt(x-a\cos\theta)\}^2 + ½\, m\,\{d/dt(a\sin\theta)\}^2 + ½\, m(r_0^2 - a^2)\dot{\theta}^2$$

$$= ½\, m(\dot{x}^2 + r_0^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\sin\theta)$$

and the potential energy

$$V = 0$$

as the weight acts through the line of hinges, that is at $(x, 0)$.

As a result, the Lagrangian

$$L = T - V = ½\, m(\dot{x}^2 + r_0^2\dot{\theta}^2 + 2a\dot{x}\dot{\theta}\sin\theta)$$

and hence the Lagrange's equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0,$$

takes the form

$$r_0^2\ddot{\theta} + a\ddot{x}\sin\theta + a\dot{x}\dot{\theta}\cos\theta = a\dot{x}\dot{\theta}\cos\theta$$

or $\ddot{\theta} = -a\ddot{x}/_{r_0^2}\sin\theta = -af/r_0^2\sin\theta$

on integration, we get

$$\dot{\theta}^2 = 2af/r_0^2\cos\theta$$

or $\dot{\theta} = -\sqrt{2af/r_0^2}\,(\cos\theta)^{½}$

as θ is decreasing with time.

Therefore, the time taken for the door to close is

$$t = -\sqrt{\frac{r_0^2}{2af}} \int_{\pi/2}^{0} d\theta/(\cos\theta)^{1/2} = \sqrt{\frac{r_0^2}{2af}} \int_{0}^{\pi/2} \frac{d\theta}{(1-2\sin^2\theta/2)^{1/2}}$$

Putting $\sqrt{2}$ sin $\theta/2$ = sin ϕ, we get

$$t = \sqrt{\frac{r_0^2}{2af}} \sqrt{2} \int_{0}^{\pi/2} \frac{\cos\phi\, d\phi}{(1-\tfrac{1}{2}\sin^2\phi)^{1/2}\,(1-\sin^2\phi)^{1/2}}$$

$$= \sqrt{\frac{r_0^2}{af}} \int^{/2} \frac{}{(1 \quad \tfrac{1}{2}\sin^2 \quad)^{1/2}} \quad \sqrt{\frac{r_0^2}{af}} \quad (\tfrac{1}{2})$$

where $E(\tfrac{1}{2})$ is elliptical integral of argument ½. Therefore,

$$t = \sqrt{\frac{r_0^2}{af}} \times 1.86$$

4.9 From equ. (9),

$$I_{xx} = \rho \int_0^L\int_0^L\int_0^L (r^2 - x^2)\, dx\, dy\, dz$$

$$= \rho \int_0^L\int_0^L\int_0^L (y^2 + z^2)\, dx\, dy\, dz$$

$$= \rho \int_0^L y^2\, dy \int_0^L dx \int_0^L dz + \rho \int_0^L z^2 dz \int_0^L dy \int_0^L dx$$

$$= 2/3\, \rho L^3.L.L$$

$$= 2/3\, ML^2$$

Similarly,

$$I_{yy} = 2/3\, ML^2 = I_{zz}$$

Because of the symmetry, all the off-diagonal elements are equal, and are given as (from equ. 10)

$$I_{xy} = -\rho \int_0^L\int_0^L\int_0^L xy\, dx\, dy\, dz$$

$$= -\tfrac{1}{4}\rho\, L^4.L$$

$$= -\tfrac{1}{4} ML^2$$

As a result, the moment of inertia tensor would be given by

$$I = ML^2 \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}$$

In order to find the principal moments of inertia, we write the determinant (or secular) equation, after substituting the values of the elements, which reads.

$$\begin{bmatrix} 2/3ML^2 - I & -1/4ML^2 & -1/4ML^2 \\ -1/4ML^2 & 2/3ML^2 - I & -1/4ML^2 \\ -1/4ML^2 & -1/4ML^2 & 2/3ML^2 - I \end{bmatrix} = 0$$

leading to a cubic equation in I. To avoid its complication, we use the fact that the value of a determinant is unaffected by adding or subtracting any row from any other row (which holds true even for columns). Subtracting the top row from the middle, we get

$$\begin{bmatrix} 2/3ML^2 - I & -1/4ML^2 & -1/4ML^2 \\ -11/12ML^2 + I & 11/12ML^2 - I & 0 \\ -1/4ML^2 & -1/4ML^2 & 2/3ML^2 - I \end{bmatrix}$$

$$= (11/12\,ML^2 - I)\begin{bmatrix} 2/3ML^2 - I & -1/4ML^2 & -1/4ML^2 \\ -1 & 1 & 0 \\ -1/4ML^2 & -1/4ML^2 & 2/3ML^2 - I \end{bmatrix}$$

$$= (11/12ML^2 - I)\,[(2/3ML^2 - I)^2 - 1/8M^2L^4 - 1/4ML^2\,(2/3ML^2 - I)] = 0$$

Solving the above equation, the three roots are given by

$$I_1 = 1/6ML^2, \quad I_2 = I_3 = 11/12ML^2$$

The Principal axis corresponding to $I_1 = 1/6ML^2$ can be found from the equation

$$\begin{bmatrix} 2/3 - 1/6 & -1/4 & -1/4 \\ -1/4 & 2/3 - 1/6 & -1/4 \\ -1/4 & -1/4 & 2/3 - 1/6 \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which simply amounts to

$$\begin{bmatrix} 1/2 & -1/4 & -1/4 \\ -1/4 & 1/2 & -1/4 \\ -1/4 & -1/4 & 1/2 \end{bmatrix}\begin{bmatrix} -0.615 \\ -0.158 \\ 0.773 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the eigen vector corresponding to $I_1 = 1/6ML^2$ is

$$\begin{bmatrix} -0.615 \\ -0.158 \\ 0.773 \end{bmatrix}$$

Similarly, corresponding to I_2 and I_3, the eigen vectors are

$$\begin{bmatrix} -0.615 \\ -0.158 \\ 0.773 \end{bmatrix} \text{ and } \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix}$$

5

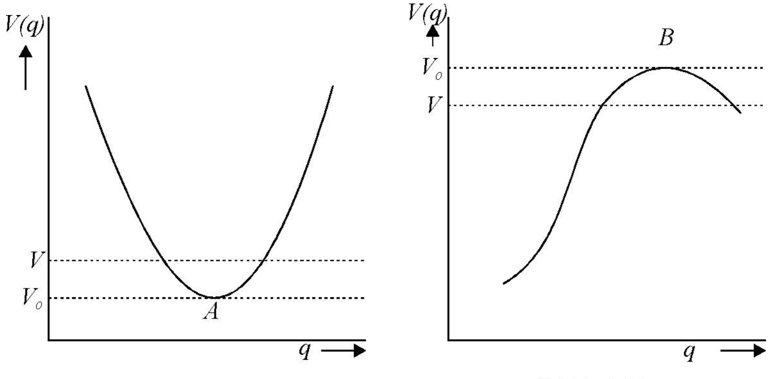
CHAPTER

Theory of Small Oscillations

5.1 INTRODUCTION

A common form of motion of mechanical systems is called small oscillations of a system about a position of the stable equilibrium. An equilibrium position is classified as stable if a small disturbance of the system from equilibrium results only in small bounded motion about the rest position. The equilibrium is said to be unstable if an infinitesimal disturbance eventually produces unbounded motion. A pendulum at rest is in stable equilibrium but an egg standing on end corresponds to an example of unstable equilibrium.

To find a relation between the potential energy V and the stability of the system, let us assume an arbitrary form of potential energy V versus q as shown in Fig. 5.1. The points A and B, where $\partial V/\partial q = 0$, are equilibrium points. Let us describe the nature of stability at these points.



(*a*) *Stable* (*b*) *Unstable*

Fig. 5.1 Shape of the potential energy curve at equilibrium

It can easily be seen that when the extremum of V is minimum (Fig. 5.1*a*), the equilibrium must be stable. Suppose the system is disturbed from equilibrium such that the potential energy changes from V_0 to V. If V is a minimum at equilibrium, any deviation from this position would produce an increase in V. By the conservation of the energy, the kinetic energy T would then decrease or effectively the velocities would decrease and come to zero or the system would come back to the equilibrium position, indicating bound motion. Thus the system is in stable equilibrium. On the other hand, if V decreases as a result of some departure from the equilibrium position (Fig. 5.1b), the kinetic energy and hence the velocities would increase indefinitely and in turn the system never returns to the equilibrium point B. Therefore, the situation corresponds to the unstable equilibrium.

The theory of small oscillations finds applications in a variety of fields such as acoustics, molecular spectra, coupled circuits etc. We shall use here the method of Lagrange's equations together with matrix tensor formulation to develop the theory of small oscillations.

5.2 FREE OSCILLATIONS IN ONE DIMENSION

We shall consider first of all the simplest case of a system with one degree of freedom. Let the equilibrium value (or position) of the generalized coordinate q be represented by q_0. A small displacement from this position results in setting up a force $-dV/dq$ which tends to return the system to equilibrium. The stable equilibrium, of course, corresponds to a position of the system at which its potential energy $V(q)$ is minimum. For small deviations from the equilibrium position

$$V(q) - V(q_0) = \tfrac{1}{2}\, k(q - q_0)^2 \qquad (1)$$

where k is a positive coefficient given by

$$k = d^2\, V(q)/dq^2\big|_{q=q_0} \qquad (2)$$

that is, second derivative of $V(q)$ at $q = q_0$. Also written as

$$k = d^2\, V(q)/dq^2\big|_0$$

Let us measure the potential energy from its minimum value, that is, put $V(q_0) = 0$ and use the symbol

$$x = q - q_0 \qquad (3)$$

for the deviation of the coordinate from its equilibrium value. Thus

$$V(x) = \tfrac{1}{2}\, kx^2 \qquad (4)$$

The kinetic energy of a system with one degree of freedom being $\tfrac{1}{2} m\dot{x}^2$, the Lagrangian of a system executing small oscillations in one dimension, would be given by

$$L = \tfrac{1}{2}\, m\dot{x}^2 - \tfrac{1}{2}\, kx^2 \qquad (5)$$

where m corresponds to the mass of the given system.

The corresponding Lagrange's equation of motion

$$d/dt\; dL/d\dot{x} - dL/dx = 0$$

reads as

$$m\ddot{x} + kx = 0$$

or

$$\ddot{x} + \omega^2 x = 0 \qquad (6)$$

where $$\omega = (k/m)^{1/2} \tag{7}$$

Equ. (6) has two independent solutions $\cos\omega t$ and $\sin\omega t$, therefore, its general solution is

$$\begin{aligned} x &= A\cos\omega t + B\sin\omega t \\ &= a\cos(\omega t + \alpha) \end{aligned} \tag{8}$$

where the arbitrary constants a and α are related to the constants A and B by

$$a = (A^2 + B^2)^{1/2},\ \text{tann}\ \alpha = -B/A \tag{9}$$

Therefore, near the position of stable equilibrium, the system executes simple harmonic oscillations. The coefficient a is called the amplitude of oscillations, and the quantity $(\omega t + \alpha)$ is called the phase of the motion. The constant α is called the phase constant (or initial value of the phase). The quantity ω is called the angular frequency of oscillations, which is a fundamental characteristic of oscillations and is independent of the initial conditions of the motion. It is entirely determined by the properties of the mechanical system (see equ. 7).

5.3 OSCILLATIONS OF SYSTEMS WITH MORE THAN ONE DEGREE OF FREEDOM

Let us consider a system with n degrees of freedom (the theory of course would be analogous to that given in sec. 5.2 for the case $n = 1$) and let its configuration be specified by generalized coordinates: $q_1, q_2, \ldots, q_n$. Furthermore, let us assume that the system is conservative, the case in which the potential energy V is function of the generalized coordinates only, that is

$$V = V(q_1, q_2, \ldots, q_n) \tag{10}$$

and also that the transformation equations defining the generalized coordinates of the system do not involve time explicitly. Thus, time dependent constraints are to be excluded from our discussion.

The system is said to be in equilibrium if the generalized forces acting on the system vanish, that is,

$$Q_i = (\partial V/\partial q_i)_0 = 0 \tag{11}$$

Zero in the suffix indicates the equilibrium configuration. The above equation implies that the potential energy V has an extremum value at the equilibrium configuration $q_{10}, q_{20}, \ldots, q_{n0}$ of the system. If the system is initially at the equilibrium position with zero initial velocities $\dot{q}_{i0}$, then the system would continue to be in equilibrium indefinitely. For example, a pendulum at rest, an egg standing on end.

Let us assume that during the motion of the system, the departures from the configuration of the stable equilibrium are small. We can, therefore, expand all the functions in a Taylor series about the equilibrium state, retaining only the lowest order terms. Let us denote the deviations of the generalized coordinates from the equilibrium by η_i and write

$$q_i = q_{io} + \eta_i, \tag{12}$$

These, q_i's may be taken as the new set of generalized coordinates of the motion. Expanding the potential energy about q_{io}, we obtain

$$V(q_1,..,q_n) = V(q_{10},...., q_{n0}) + \sum_i(\partial V/\partial q_i)_0\,\eta_i + \tfrac{1}{2}{}_{ij}(\partial^2V/\partial q_i\partial q_j)_0\eta_i\eta_j + ... \tag{13}$$

The terms linear in η_i vanish automatically because of the equilibrium condition (11). The first term in the series is the potential energy of the equilibrium position and by shitting the arbitrary zero of the potential energy to coincide with the potential energy at equilibrium, this term can also made to vanish. We are, therefore, left with the quadratic terms as the first approximation to V. Thus

$$V = \tfrac{1}{2}\sum_{ij}(\partial^2V/\partial q_i\partial q_j)_0\eta_i\eta_j = \tfrac{1}{2}\sum_{ij}V_{ij}\eta_i\eta_j \tag{14}$$

where the 2nd derivatives of V have been designated by V_{ij} depending only on the equilibrium values of the q_i's, that is, on q_{i0}'s. It is obvious from their definition that the V_{ij}'s are symmetrical, that is, $V_{ij} = V_{ji}$ as the individual terms in the above equation are unaffected by interchanging the indices.

Further since the generalized coordinates do not involved time explicitly, the kinetic energy is a homogenous quadratic function of the generalized velocities, that is

$$T = \tfrac{1}{2}\sum_{ij} m_{ij}\,\dot{q}_i\dot{q}_j = \tfrac{1}{2}\sum_{ij} m_{ij}\,\dot{\eta}_i\dot{\eta}_j \tag{15}$$

as $\dot{q}_{i0} = \dot{q}_{j0} = 0$. The coefficients.

$$m_{ij} = \sum_k m_k \frac{\partial\vec{r}_k}{\partial q_i}\cdot\frac{\partial\vec{r}_k}{\partial q_j}$$

are in general functions of the coordinates q_k and as before they may be expanded in a Taylor series about the equilibrium configuration, yielding

$$m_{ij}(q_{1,\,........,}\,q_n) = m_{ij}\,(q_{10},\,........., q_{n0}) + \sum_k\left(\frac{\partial m_{ij}}{\partial q_k}\right)_0\eta_k + ...$$

Since T as given by equ. (15) is already quadratic in the η_i's, we can get the lowest non vanishing approximation to T by dropping all the terms except the first one in the expansion of m_{ij}, that is, by taking

$$m_{ij} \approx m_{ij}\,(q_{10},\,......., q_{n0})$$

Denoting the constant values of m_{ij}, that is $m_{ij}\,(q_{10},....q_{n0})$ functions at equilibrium by T_{ij}, we can write the kinetic energy as

$$T = \tfrac{1}{2}\sum_{ij} T_{ij}\,\dot{\eta}_i\dot{\eta}_j \tag{16}$$

where the constants are again symmetric since the individual terms in the above equation are unaffected by an interchange of indices.

From equs. (14) and (16), the Lagrangian of the system executing small oscillations, is given by

$$L = T - V = \tfrac{1}{2}\sum_{ij}(T_{ij}\,\dot{\eta}_i\dot{\eta}_j - V_{ij}\,\eta_i\eta_j) \tag{17}$$

Taking η's as the generalized coordinates,

$$\partial L/\partial \dot{\eta}_i = \tfrac{1}{2}\sum_i T_{ij}\,\dot{\eta}_j,\ \partial L/\partial \eta_i = -\tfrac{1}{2}\sum_j V_{ij}\,\eta_j$$

and hence the Lagrange's equations of motion are given by

$$\sum_j (T_{ij}\ddot{\eta}_j + V_{ij}\,\eta_j) = 0,\ i = 1, 2, \ldots\ldots\ldots., n \tag{18}$$

which form a set of a n homogeneous simultaneous linear differential equations with constant coefficients. It is this set of equations which must be solved to obtain the motion near the equilibrium.

Since we are dealing with the oscillatory motion, we are, therefore lead to try an oscillatory solution of the form

$$\eta_j = a_j e^{i\omega t} \tag{19}$$

where a_j gives the amplitude of oscillation corresponding to each coordinate η_j and is to be determined. Such solutions are called normal modes of oscillation of the system.

Substituting the trial solution (19) into the equations of motion (18) leads to a set of linear homogeneous algebraic equations to be satisfied by a_j, that is

$$\sum_j (-\omega^2\, T_{ij} + V_{ij})a_j = 0 \tag{20}$$

If this system has non – vanishing solutions, the determinant of the coefficients must vanish, that is

$$|V_{ij} - \omega^2\, T_{ij}| = 0 \tag{21}$$

which amounts to

$$\begin{vmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} \ldots\ldots\ldots\ldots V_{1n} - \omega^2 T_{1n} \\ V_{21} - \omega^2 T_{21} & V_{22} - \omega^2 T_{22} \ldots\ldots\ldots\ldots V_{2n} - \omega^2 T_{2n} \\ V_{31} - \omega^2 T_{31} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ V_{n1} - \omega^2 T_{n1} & V_{n2} - \omega^2 T_{n2} \ldots\ldots\ldots\ldots V_{nn} - \omega^2 T_{nn} \end{vmatrix} = 0 \tag{22}$$

This is the characteristic or secular equation of degree n in ω^2. In general, it has n different real positive roots $\omega_\alpha^2(\alpha = 1, 2, \ldots\ldots, n)$. In particular cases, some of these roots may coincide or may be equal, the phenomenon is called degeneracy. The quantities ω_α thus determined are called the characteristic or eigen frequencies of the system. It is evident that the roots of equ. (22) are real and positive. For the existence of an imaginary part of ω would mean the presence, in the time dependence of the coordinates η_j, of an exponentially decreasing or increasing factor. Such a factor is inadmissible as it would lead to a time variation of the total energy $E = T + V$ of the system which would, therefore, not be conserved.

The frequencies ω_α having been found, we substitute each of them in equs. (20) and find the corresponding coefficients a_j. Since there exist n values of ω_α, we can form n sets of the values of a_j. Each of these sets, that is, n values of a_j corresponding to the value of ω_α, can be considered to define the components of n – dimensional vector $\boldsymbol{a}_\alpha$. This vector $\boldsymbol{a}_\alpha$ is called eigen vector of the system. The symbol a_{jr} can be used to represent the *jth* component of the *rth* eigen vector.

To complete the discussion, it may be pointed out that the n homogeneous simultaneous linear differential equs. (18) can be considered as the n components of the matrix equation

$$(T)\,(\ddot{\eta}) + (V)\,(\eta) = 0, \tag{23}$$

where the matrices (T), (V), $(\ddot{\eta})$, and (η) are given by

$$(T) = \begin{bmatrix} T_{11} & T_{12}\ldots\ldots\ldots\ldots T_{1n} \\ T_{21} & T_{22}\ldots\ldots\ldots\ldots T_{2n} \\ \vdots & \\ T_{n1} & T_{n2}\ldots\ldots\ldots\ldots T_{nn} \end{bmatrix}, \tag{24}$$

$$(V) = \begin{bmatrix} V_{11} & V_{12}\ldots\ldots\ldots\ldots V_{1n} \\ V_{21} & V_{22}\ldots\ldots\ldots\ldots V_{2n} \\ \vdots & \\ V_{n1} & V_{n2}\ldots\ldots\ldots\ldots V_{nn} \end{bmatrix}, \tag{25}$$

$$(\ddot{\eta}) = \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \\ \vdots \\ \ddot{\eta}_n \end{bmatrix} = \frac{d^2}{dt^2}\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}, \tag{26}$$

and

$$(\eta) = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} \tag{27}$$

Equ. (23) may in turn be considered to be the matrix representation of the operator equation

$$T\,|\,\ddot{\eta} > + V\,|\,\eta > = 0 \tag{28}$$

where $|\,\eta >$ is the n – dimensional vector whose matrix representation is (η), and T and V are two operators having the matrix representations (T) and (V).

5.4 NORMAL COORDINATES

Let us consider two harmonic oscillators (masses) coupled together by a spring of strength k. Each one is again joined to fixed wall by spring of strength k (see Fig. 5.2). The motion of the two masses is constrained to take place along a straight line. Let x_1 and x_2 represent the displacements of the two masses from the equilibrium positions O_1 and O_2, respectively. The kinetic energy of the system is given by

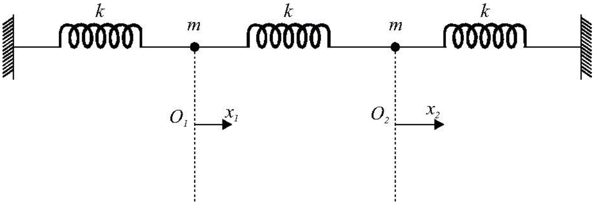


Fig. 5.2 Two equal masses coupled by identical springs to each other and to two fixed walls.

$$T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 \tag{29}$$

and the potential energy of the system measured from the equilibrium positions is

$$V = \tfrac{1}{2}k x_1^2 + \tfrac{1}{2}k(x_2 - x_1)^2 + \tfrac{1}{2}k x_2^2 \tag{20}$$

Therefore, the Lagrangian L of the system is

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 - \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 - \tfrac{1}{2}k(x_2 - x_1)^2 \end{aligned} \tag{31}$$

and as a result, the Lagrange's equations of motion

$$d/dt\,(\partial L/\partial \dot{x}_1) - \partial L/\partial x_1 = 0$$

and $$d/dt(\partial L/\partial \dot{x}_2) - \partial L/\partial x_2 = 0$$

take the forms

$$m\ddot{x}_1 + kx_1 - k(x_2 - x_1) = 0 \tag{32a}$$

and $$m\ddot{x}_2 + kx_2 + k(x_2 - x_1) = 0 \tag{32b}$$

To obtain the possible modes of vibrations, we must solve the above second order differential equations. To do so, we assume trial solutions of the forms

$$x_1 = Ae^{i\omega t} \text{ and } x_2 = B\,e^{i\omega t} \tag{33}$$

Substituting these in equs. (32), we obtain after rearranging

$$(2k - m\omega^2)\,A - kB = 0 \tag{34a}$$

and $$-kA + (2k - m\omega^2)B = 0 \tag{34b}$$

Thus we have two homogeneous simultaneous differential equations with three unknowns A, B and ω. These equations can be solved to give

$$A/B = \frac{k}{2k - m\omega^2} = \frac{2k - m\omega^2}{k} \tag{34c}$$

To solve the equs. (34a, b) for ω, we would use the fact that the said equations would have a solution only if the determinant of the coefficients of A and B vanishes, that is,

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0 \tag{35}$$

called the secular equation. This amounts to

$$(2k - m\omega^2)^2 - k^2 = 0$$

or

$$(2k - m\omega^2 - k)\,(2k - m\omega^2 + k) = 0 \tag{36}$$

which gives the following two roots,

$$\omega = \pm\omega_1 = \pm(\mathrm{k/m})^{1/2} \tag{37a}$$

and

$$\omega = \pm\omega_2 = \pm(3\mathrm{k/m})^{1/2} \tag{37b}$$

As a result, the general solutions of equs. (32) may be written as

$$x_1 = A_1 e^{i\omega_1 t} + A_{-1}\, e^{-i\omega_1 t} + \mathrm{A}_2 e^{i\omega_2 t} + A_{-2} e^{-i\omega_2 t} \tag{38a}$$

$$x_2 = B_1 e^{i\omega_1 t} + B_{-1}\, e^{-i\omega_1 t} + B_2 e^{i\omega_2 t} + B_{-2} e^{-i\omega_2 t} \tag{38b}$$

Substituting equs. (37) in equs. (34), we can obtain the ratios A/B for different values of ω to be

If $\quad \omega = \omega_1, \quad A = B \quad (39a)$

If $\quad \omega = \omega_2, \quad A = -B \quad (39b)$

Combining equs. (39) with equs. (38), we obtain

$$x_1 = A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t} + \mathrm{A}_2 e^{i\omega_2 t} + \mathrm{A}_{-2} e^{-i\omega_2 t} \tag{40a}$$

$$x_2 = A_1 e^{i\omega_2 t} + A_{-1} e^{-i\omega_1 t} - \mathrm{A}_2 e^{i\omega_2 t} - \mathrm{A}_{-2} e^{-i\omega_2 t} \tag{40b}$$

where A_1, A_{-1}, A_2, A_{-2} are the constants. After determining the constants from initial conditions, one notices that the each coordinate (x_1 or x_2) depends on two frequencies ω_1 and ω_2. Therefore, it may not be a simple task to interpret the type of motion with which the system is oscillating. It is possible to find new coordinates X_1 and X_2, which are linear combinations of x_1 and x_2, such that each coordinate oscillates with a single frequency. Use of equs. (40) give the new coordinates as the sum and difference of x_1 and x_2, that is

$$\begin{aligned} X_1 = x_1 + x_2 &= 2(A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t}) \\ &= Ce^{-i\omega_1 t} + De^{-i\omega_1 t} \end{aligned} \tag{41a}$$

$$\begin{aligned} X_2 = x_1 - x_2 &= 2(A_2 e^{i\omega_2 t} + A_{-2} e^{-i\omega_2 t}) \\ &= Ee^{i\omega_2 t} + Fe^{-i\omega_2 t} \end{aligned} \tag{41b}$$

where C, D, E, and F are new constants.

The coordinates X_1 and X_2 correspond to the modes of oscillation involving only one frequency ω_1 or ω_2. These are called normal modes of oscillations and the corresponding coordinates are called the normal coordinates.

The nature of any one of the normal modes can be investigated if all other normal modes can be equated to zero. Suppose that $X_2 = 0$, one can study the motion represented by X_1 mode. Then

$$x_1 - x_2 = 0$$

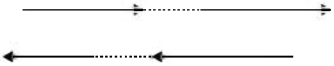
or $$x_1 = x_2 \tag{42}$$

Thus X_1 is a symmetric mode as shown in Fig. 5.3(a). Both masses have equal displacements, both oscillate with the same frequency ω_1 and are in phase.

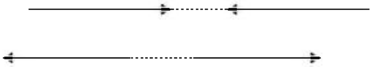
Now suppose $X_1 = 0$, in that case one can study the motion represented by X_2 mode. Then $$x_1 + x_2 = 0$$

or $$x_1 = -x_2 \tag{43}$$

Thus X_2 is an antisymmetric mode as shown in Fig. 5.3(b). Both masses have equal and opposite displacement or move out of phase with the same frequency ω_2.



Symmetrical mode X_1:
$X_2 = 0$ and $x_1 = x_2$
(a)



Antisymmetrical mode X_2:
$X_1 = 0$ and $x_1 = -x_2$
(b)

Fig. 5.3 Modes of vibration of two coupled masses (oscillators) (a) symmetric mode (b) antisymmetric mode.

From the above discussion it is clear that in the symmetric mode, the two masses vibrate as if there were no coupling between them (the spring is neither compressed nor elongated), and their frequency is the same as the original frequency $\omega_1 = (k/m)^{1/2}$. In the antisymmetric mode, the two masses oscillate out of phase and their frequency is higher ($\omega_2 = (3k/m)^{1/2}$ than their individual uncoupled frequency.

Let us now try to obtain the equations of motion in terms of the normal coordinates. To do so, we find from equs. (41),

$$x_1 = \frac{X_1 + X_2}{2} \tag{44}$$

and $$x_2 = \frac{X_1 - X_2}{2} \tag{45}$$

Substituting the above values in equs. (32), we get

$$\frac{m}{2}\left(\ddot{X}_1 + \ddot{X}_2\right) + k(X_1 + X_2) + kX_2 = 0$$

and $$-\left(\ddot{X}_1 \quad \ddot{X}_2\right) + k(X_1 - X_2) - kX_2 = 0$$

Adding and substracting the above equations, we get

$$m\ddot{X}_1 + kX_1 = 0$$

or $$\ddot{X}_1 + k/m\, X_1 = 0 \tag{46a}$$

$$m\ddot{X}_2 + 3kX_2 = 0$$

or
$$\ddot{X}_2 + 3k/m\,X_2 = 0 \tag{46a}$$

That is, the X_1 mode vibrates with frequency $\omega_1 = (k/m)^{1/2}$ and the X_2 mode vibrates with frequency $\omega_2 = (3k/m)^{1/2}$ which is in agreement with the result derived earlier. In other words this simply amounts to the conclusion that there is one differential equation for each normal coordinate and hence the solution of each differential equation represents a separate mode of vibration.

5.5 NORMAL COORDINATES – GENERAL PROCEDURE

Consider a system with n degrees of freedom and having small oscillations about equilibrium positions. We take here the generalized coordinates as q'_1, q'_2, ... q'_n whereas the equilibrium configuration is described by the coordinates q'_{10}, q'_{20}, ..., q'_{n0}. As a result the potential energy of the system in the present case can be written as

$$V = ½\sum_i^n \sum_j^n V_{ij}\eta'_i\eta'_j$$

where
$$\eta'_i = q'_i - q'_{i0}, \quad \text{and} \quad V_{ij} = (\partial^2 V/\partial\, q'_i q'_j)_0$$

For such a system, V may be written as

$$V = a_{11}\eta_1'^2 + a_{22}\eta_2'^2 + ... + a_{nn}\eta_n'^2 + 2a_{12}\eta'_1\eta'_2 + ...$$

in which every term is quadratic in the coordinates and coefficients a_{11}, a_{22}, ..., a_{nn}, a_{12},.... are constants. There are however, square and cross terms present in V.

In the similar fashion, if the kinetic energy T does not contain time explicity, it would be a homogeneous quadratic function of velocities and may be written as

$$T = b_{11}\dot{\eta}_1'^2 + b_{22}\dot{\eta}_2'^2 + ... + b_{nn}\dot{\eta}_n'^2 + b_{12}\dot{\eta}'_1\dot{\eta}'_2 + ...$$

where b_{11}, b_{22}, ...,b_{nn}, b_{12}, ... are constants. There are once again square and cross terms present.

It is possible to cause a linear transformation to new generalized coordinates η_1, η_2,........, η_n with the help of the linear combinations of the coordinates η'_1, η'_2,.............., η'_n such that

$$\eta_1 = e_{11}\eta'_1 + e_{12}\eta'_2 + \text{..........} + e_{1n}\eta'_n$$
$$\eta_2 = e_{21}\eta'_1 + e_{22}\eta'_2 + \text{..........} + e_{2n}\eta'_n$$
$$\eta_n = e_{n1}\eta'_1 + e_{n2}\eta'_2 + \text{..........} + e_{nn}\eta'_n$$

In such a situation, V and T do not contain cross terms and would take the following forms,

$$V = ½(\lambda_1\eta_1^2 + \lambda_2\eta_2^2 + \text{...............} + \lambda_n\eta_n^2)$$

and
$$T = ½(m_1\dot{\eta}_1^2 + m_2\dot{\eta}_2^2 + \text{...............} + m_n\dot{\eta}_n^2)$$

where λ's and m's are constants.

As a result, the Lagrange's equations of motion would read as

$$\ddot{\eta}_n + \omega_n^2 \eta_n = 0$$

in which the coordinates $\eta_1, \eta_2, \ldots, \eta_n$ are termed as the normal coordinates and $\omega_1, \omega_2, \ldots, \omega_n$ the corresponding normal frequencies. As each normal coordinate varies with only one normal frequency and hence these are called normal modes of vibration.

5.6 SMALL TRANSVERSE OSCILLATIONS OF PARTICLES ON A STRING

Consider a light string stretched with force F(or stretched to a tension F) with n equal masses m placed along it at regular intervals l (see Fig. 5.4). We shall consider small transverse oscillations of the particles and use the displacements $y_1, y_2, \ldots,$ y_n from the equilibrium positions as our generalized coordinates. Therefore, the kinetic energy of the system.

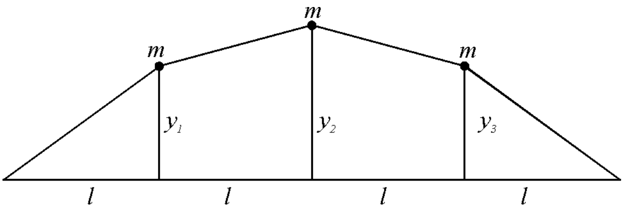


Fig. 5.4 Particles on a string shown displaced from their equilibrium positions

$$T = \tfrac{1}{2} m \left(\dot{y}_1^2 + \dot{y}_2^2 + \ldots\ldots + \dot{y}_n^2 \right) \tag{47}$$

To calculate the potential energy, let us consider the length of the string between j^{th} and $(j + 1)$th particles. In equilibrium, its length is l, but when the particles are displaced, it is

$$l + \delta l = [l^2 + (y_{j+1} - y_j)^2]^{1/2}$$

$$= l\,[1 + (y_{j+1} - y_j)^2/2l^2]$$

or
$$\delta l = (y_{j+1} - y_j)^2/2l \tag{48}$$

where we have neglected the higher order terms in the expansion of the quantity under the radical sign. Note that the above relation applies also to the sections of the string at each end if we assume that $y_0 = y_{n+1} = 0$.

The work done against the tension in increasing the length of the piece of string by this amount is $F\delta l$. Thus, adding the contributions from each piece of string (or the total work done in displacing the mass points from their equilibrium positions), we find that the potential energy is given by (see equ. 48)

$$V = F/2l[y_1^2 + (y_2 - y_1)^2 + \ldots\ldots + (y_n - y_{n-1})^2 + y_n^2] \tag{49}$$

As a result, the Lagrangian of the system is

$$L = \tfrac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2 + \ldots\ldots + \dot{y}_n^2)$$
$$- F/2l[y_1^2 + (y_2 - y_1)^2 + \ldots\ldots + (y_n - y_{n-1})^2 + y_n^2]$$

and therefore, the Lagrange's equations of motion are given by

$$\begin{aligned} m\ddot{y}_1 + F/l(2y_1 - y_2) &= 0, \\ m\ddot{y}_2 + F/l(-y_1 + 2y_2 - y_3) &= 0 \\ &\ldots\ldots \\ m\ddot{y}_n + F/l(-y_{n-1} + 2y_n) &= 0 \end{aligned} \tag{50}$$

Writing $F/ml = \omega_0^2$ and then substituting the solutions

$$y_j = a_j e^{i\omega t}$$

into equs. (50), we get the following equations

$$\begin{aligned} (2\omega_0^2 - \omega^2)a_1 - \omega_0^2 a_2 &= 0, \\ -\omega_0^2 a_1 + (2\omega_0^2 - \omega^2)a_2 - \omega_0^2 a_3 &= 0, \\ &\ldots\ldots \\ -\omega_0^2 a_{n-1} + (2\omega_0^2 - \omega^2)a_n &= 0 \end{aligned} \tag{51}$$

For $n = 1$, we have only one normal mode with frequency

$$\omega^2 = 2\omega_0^2$$

For $n = 2$, the characteristics equation is

$$\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0$$

which amounts to

$$(2\omega_0^2 - \omega^2)^2 - \omega_0^4 = 0$$

and we have two normal modes

$$\omega^2 = \omega_1^2 = \omega_0^2,\ a_1/a_2 = 1$$

and
$$\omega^2 = \omega_2^2 = 3\omega_0^2,\ a_1/a_2 = -1$$

For $n = 3$, the characteristic equation is given by

$$\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{vmatrix} = 0$$

Expanding the determinant, we get the following cubic equation in ω^2,
$(2\omega_0^2 - \omega^2)^3 - 2\,\omega_0^4(2\omega_0^2 - \omega^2) = 0$

The roots of the above equation are $2\omega_0^2$ and $(2\pm\sqrt{2})\omega_0^2$. Accordingly, the corresponding normal modes are

$\omega^2 = \omega_1^2 = (2 - \sqrt{2})\omega_0^2$, $a_1/a_2 = 1/\sqrt{2} = a_3/a_2$

$\omega^2 = \omega_2^2 = 2\omega_0^2$, $a_1/a_3 = -1$, $a_2 = 0$

$\omega^2 = \omega_3^2 = (2 + \sqrt{2})\omega_0^2$, $a_1/a_2 = -1/\sqrt{2} = a_3/a_2$

In the similar fashion, one can obtain normal modes for $n = 4, 5, \ldots\ldots$. It may be noted here that for every value of n, the slowest mode is the one in which all the masses are oscillating in the same direction, while in the fastest mode alternate masses oscillate in opposite directions (see Fig. 5.5)

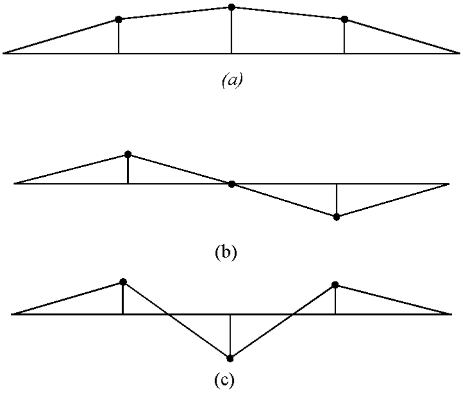


Fig. 5.5 Modes of vibration for (*a*) $n = 1$, (*b*) $n = 2$, and (*c*) $n = 3$.

PROBLEMS

5.1 Find ratio of the two frequencies ω and ω' of the oscillations of two diatomic molecules consisting of atoms of different isotopes, the masses of atoms being m_1, m_2, and m'_1, m'_2.

5.2 Find frequency of oscillations of a particle of mass m which is free to move along a line and is attached to a spring whose other end is fixed at a point A (Fig. 5.7) at a distance l from the line. A force F is required to extend the spring.

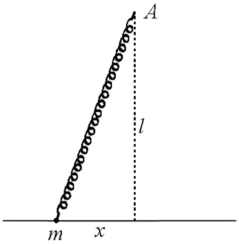


Fig. 5.7 A particle of mass m is free to move along a line and is attached to a spring whose end A is fixed.

5.3 Find frequency of oscillations of a particle of mass m moving freely on a circle of radius r (seed Fig. 5.8) and is attached to a spring whose other end is fixed at A.

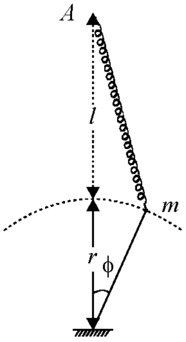


Fig. 5.8 A particle of mass m moving freely on a circle of radius r and is attached to a spring whose end A is fixed.

5.4 Find the frequency of oscillations of a simple pendulum of mass m_2 whose point of support carries a mass m_1 which is free to move on a horizontal line lying in the plane in which m_2 moves (see Fig. 1.9).

5.5 Find the frequencies of small oscillations in the case of a double pendulum vibrating in a vertical plane (see Fig. 1.7). Assume that $m_1 = m_2 = m$ and $l_1 = l_2 = l$.

5.6 Obtain the normal modes of vibration for the double pendulum shown in Fig. 1.7, assuming equal lengths, but not equal masses. Show that when the lower mass is small compared to the upper one ($m_2/m_1 \to 0$, $m_1 + m_2 \approx m_1$), the two resonant frequencies are almost equal.

5.7 Consider two identical simple pendulums each having length l and a bob of mass m coupled to each other by means of a horizontal spring of spring constant k (see Fig. 5.9). Find the frequencies of the normal modes and the corresponding eigen vectors.

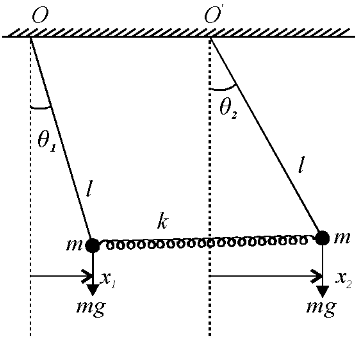


Fig. 5.9 Two coupled simple pendulums.

5.8 Two unit masses are coupled by springs of unit strength to a mass m as shown in Fig. 5.10. Investigate the small longitudinal oscillations.

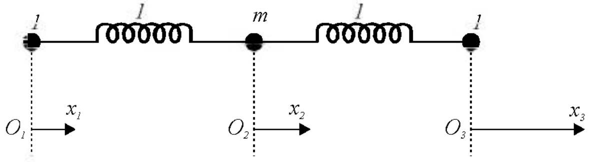


Fig. 5.10 Two unit masses coupled by springs of unit strength to a mass m.

5.9 Two particles, each of unit mass, are connected by a spring of strength k to each other and by springs of unit strength to two fixed supports as shown in Fig. 5.11. The whole system is lying in a straight line. Find the frequencies of small oscillations and the corresponding eigen vectors.

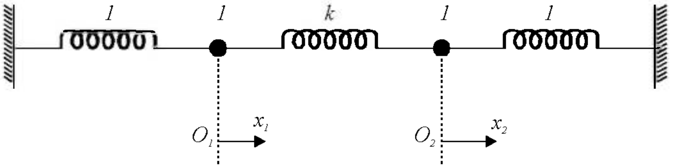


Fig. 5.11 Two particles, each of unit mass coupled together by a spring of strength k and connected by springs of unit strength to two fixed supports.

5.10 A typical triatomic molecule may be regarded as equivalent to two atoms of mass m each symmetrically located on each side of an atom of mass M (See Fig. 5.12). The potential energy of the system is approximated by two springs of strength k joining the three atoms. Assume that the vibrations are along the line joining the three masses. Find the possible modes of vibration.

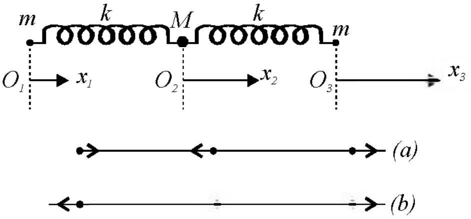


Fig. 5.12 Model of a linear symmetric triatomic molecule (a) Normal mode corresponding to X_1(*b*) Normal mode corresponding to X_2.

5.11 Obtain frequencies of the normal modes of a symmetrical triatomic molecule shown in Fig. 5.12. Reduce the problem to one of the two degrees of freedom by introducing the new coordinates $X_1 = x_1 + x_3, X_2 = x_1 - x_3$ and eliminating x_2 by requiring that the centre of mass of the system remains at rest.

5.12 Find the frequencies of oscillations of a system with two degrees of freedom whose Lagrangian is given by

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \tfrac{1}{2}\omega_0^2(x^2 + y^2) + \alpha\, x\, y$$

which corresponds to two identical one – dimensional systems of characteristic frequency ω_0 coupled by an interaction $-\alpha\, x\, y$. Find equations of motion in terms of the normal coordinates.

5.13 A typical diatomic molecule may be considered as equivalent to two masses m_1 and m_2 joined by a massless spring of strength k. Assume the vibrations to be confined along the line joining the two masses (Fig. 5.13). Find the normal modes and their frequencies.

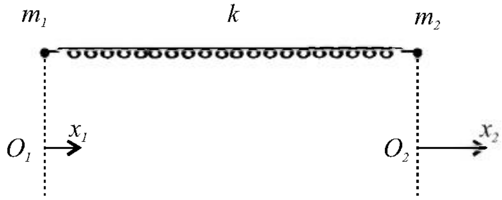


Fig. 5.13 Two masses m_1 and m_2 connected by a spring of strength k and vibrating along the line joing the two masses.

5.14 Consider a triangle, with each side a, rotating at a constant angular velocity ω about vertical axis (see Fig. 5.16). The mass m slides without friction on one leg of the triangle and is fastened to a spring whose equilibrium lengths is l and whose spring constant is k. Find the frequency with which the mass m undergoes small oscillations.

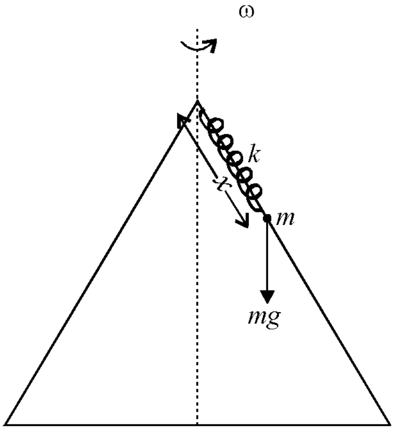


Fig. 5.16 A triangle rotating at a constant angular velocity ω with mass m fastened to a spring and sliding without friction on one leg of the triangle.

5.15 A mass M is suspended from a ceiling by a spring of force constant K. A second mass m is suspended from the first by a spring of force constant k

(see Fig. 5.17). Neglecting spring masses and assuming a uniform gravitational field of acceleration g, find the equations of motion and the frequencies of the normal modes of vibration.

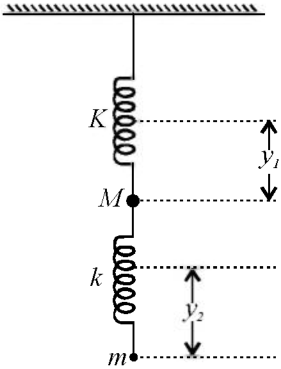


Fig. 5.17 Mass M suspended by a spring of constant K. A second mass m suspended from the first by spring of constant k.

5.16 A bead of mass m slides without friction on a uniform circular ring of mass m and radius a. The ring oscillates under gravity in its own plane about a fixed point on its circumference, find the oscillation frequencies and the eigen vector components (Fig. 5.18).

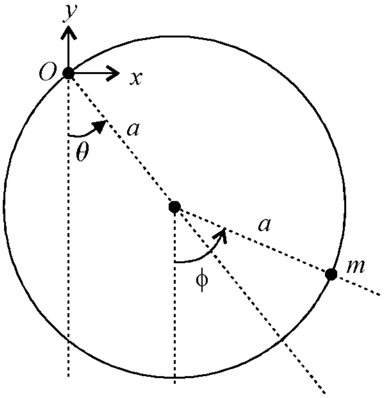


Fig. 5.18 A bead sliding on a uniform circular ring.

SOLUTIONS

5.1 Let us consider a diatomic molecule which can oscillate along its axis of symmetry. The coupling between the atoms that makes up the molecule is electromagnetic but we would consider here the equivalent mechanical problem in which the two atoms of masses m_1 and m_2 are connected by a massless spring of force constant k (see Fig. 5.6) and the system is free to oscillate on a frictionless horizontal surface. We locate the ends of the spring by the coordinates $x_1(t)$ and $x_2(t)$ as shown in Fig. 5.6(*a*). The length of the spring at any instant is, therefore, $x_1 - x_2$. The change in the length of the spring $x(t)$ is given by

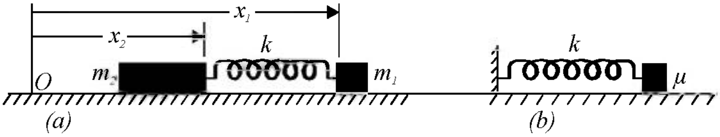


Fig. 5.6 (*a*) Two bodies of masses m_1 and m_2 connected by a mass less spring of unstretched length *l* (*b*) a single body of mass μ connected by an identical spring to a rigid wall.

$$x = (x_1 - x_2) - l \tag{52}$$

where *l* is the unstretched length of the spring.

The kinetic energy of the system is given by

$$T = ½m_1\dot{x}_1^2 + ½m_2\dot{x}_2^2$$

and the potential energy

$$V = ½k(x_1 - x_2 - l)^2$$

As a result, the Lagrange's equations of motion would be given by

$$m_1\ddot{x}_1 - kx = 0 \tag{53a}$$

and $$m_2\ddot{x}_2 + kx = 0 \tag{53b}$$

Multiplying equ. (53*a*) by m_2 and (53*b*) by m_1 and then subtracting, we obtain

$$\frac{m_1m_2}{m_1 + m_2} d^2(x_1 - x_2)/dt^2 = -kx \tag{54}$$

l being a constant, $d^2(x_1 - x_2)/dt^2 = d^2/dt^2\{(x_1 - x_2) - l\} = d^2x/dt^2$.

Further writing $$\mu = m_1m_2/(m_1 + m_2) \tag{55}$$

called the reduced mass of the system, equ. (54) takes the form

$$d^2x/dt^2 + k/\mu\, x = 0 \tag{56}$$

Thus, it is clear that the present system has the same frequency as that of a single mass μ, connected by a similar spring to a rigid wall (see Fig. 5.6(*b*). Hence, the two body oscillation of a Fig. 5.6(*a*) is equivalent to the one body oscillation of Fig. 5.6(*b*).

As an outcome, the atoms of the diatomic molecule oscillate with frequency

$$\omega = (k/\mu)^{1/2} \tag{57a}$$

Since the atoms of the isotopes interact in the same way, we have $k = k'$.

Accordingly, the frequency of oscillations of a diatomic molecules with atoms having masses m_1' and m_2' would be given by

$$\omega' = (k/\mu')^{1/2}$$

where $$\mu' = m_1' m_2'/(m_1' + m_2')$$

Therefore, $$\omega'/\omega = \left[\frac{m_1 m_2 (m_1' + m_2')}{m_1' m_2' (m_1 + m_2)}\right]^{1/2}$$

5.2 The potential energy of the spring is given by

$$V = F\,\delta l$$

where δl denotes the extension of the spring.

For $$x << l,\ \delta l = (l^2 + x^2)^{1/2} - l = l\,(1 + x^2/l^2)^{1/2} - l$$

$$= l\,(1 + x^2/2l^2) - l = x^2/2l.$$

Therefore, $$V = F\,x^2/2l$$

The kinetic energy being

$$T = ½\, m\dot{x}^2,$$

the Lagrangian of the system would be given by

$$L = T - V = ½\, m\dot{x}^2 - Fx^2/2l$$

As a result, the Lagrange's equation of motion,

$$d/dt\ dL/d\dot{x} - dL/dx = 0$$

would take the form

$$m\ddot{x} + Fx/l = 0$$

or $$\ddot{x} + F/ml\ x = 0$$

Therefore, the frequency of oscillations of the particle of mass m in the present case would be given by

$$\omega = (F/ml)^{1/2}$$

5.3 The extension of the spring in the present case is

$$\delta l = [r^2 + (l + r)^2 - 2r(l + r)\cos\phi]^{1/2} - l$$

where l is the unstretched length of the spring.

For ϕ small, $$\delta l = [r^2 + (l + r)^2 - 2r(l + r)\,(1 - \phi^2/2]^{1/2} - l$$

$$= [l^2 + r(l + r)\,\phi^2]^{1/2} - l$$

$$= l\,[1 + r(l + r)/l^2\,\phi^2]^{1/2} - l$$

$$= l\,[1 + r(l + r)/2l^2\,\phi^2] - l$$

$$= r(l + r)/2l\,\phi^2$$

Therefore, the potential energy of the spring

$$V = F\delta l = Fr(l + r)/2l\,\phi^2$$

where F is the force required to extend the spring.

The kinetic energy in the present case is given by

$$T = \tfrac{1}{2}\,mr^2\dot{\phi}^2$$

Therefore, the Lagrangian of the system

$$L = T - V = \tfrac{1}{2}\,mr^{20}\dot{\phi}^2 - Fr(l + r)/2l\,\phi^2$$

and hence the Lagrange's equation of motion would read

$$mr^2\ddot{\phi} + Fr(l + r)/l\,\phi = 0$$

or $$\ddot{\phi} + F(l + r)/mrl\,\phi = 0$$

Thus, the frequency of oscillations of the particle of mass m moving freely on the circle would be given by

$$\omega = [F(l + r)/mrl]^{1/2}$$

5.4 The coordinates of mass m_1 are, $x_1 = x$, $y_1 = 0$ and that of mass m_2 are $x_2 = x + l\sin\phi$, $y_2 = l\cos\phi$. Therefore, the kinetic energy of the system

$$\begin{aligned} T &= \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2\,[(\dot{x} + l\cos\phi\dot{\phi})^2 + l^2\sin^2\phi\dot{\phi}^2] \\ &= \tfrac{1}{2}(m_1 + m_2)\,\dot{x}^2 + \tfrac{1}{2}m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) \end{aligned}$$

The potential energy in this case is

$$V = -\,m_2\,gl\cos\phi$$

Therefore, the Lagrangian L is given by

$$\begin{aligned} L = T - V = \tfrac{1}{2}(m_1 + m_2)\,\dot{x}^2 + \tfrac{1}{2}\,m_2(l^2\dot{\phi}^2 + 2l\dot{x}\dot{\phi}\cos\phi) \\ + m_2\,gl\cos\phi \end{aligned}$$

As x does not appear in the Lagrangian, x is cyclic. Hence the conjugate momentum p_x, which is the horizontal component of the total momentum of system, is conserved. That is,

$$p_x = \partial L/\partial\dot{x} = (m_1 + m_2)\,\dot{x} + m_2 l\dot{\phi}\cos\phi = \text{constt.}$$

or

The system may always be taken to be at rest as a whole. Then the constant in the above equation is zero, that is,

$$(m_1 + m_2)\,\dot{x} + m_2 l\dot{\phi}\cos\phi = 0$$

or $$\dot{x} = -\frac{m_2 l\,\dot{\phi}\cos\phi}{(m_1 + m_2)}$$

Substituting the value of $\dot{x}$ from the above expression into the expression for the Lagrangian, one gets

$$L = ½(m_1 + m_2)\frac{m_2^2 l^2\,\dot\phi^2 \cos^2\phi}{(m_1 + m_2)} + ½m_2 l^2\dot\phi^2$$

$$+ m_2\, l\,\dot\phi \cos\phi \times \frac{-m_2 l\,\dot\phi \cos\phi}{m_1 + m_2} + m_2 g l \cos\phi$$

$$= ½\, m_2 l^2\, \dot\phi^2 - ½\frac{m_2^2 l^2\, \dot\phi^2 \cos^2\phi}{(m_1 + m_2)} + \mathrm{m}_2\, \mathrm{g}\, l \cos\phi$$

$$= ½\,\frac{m_1 m_2 l^2\, \dot\phi^2}{m_1 + m_2} + ½\frac{m_2^2 l^2\, \dot\phi^2}{m_1 + m_2}\,(1 - \cos^2\phi)$$
$$+ m_2\, g\, l\,(1 - \phi^2/2)$$

For small ϕ,
$$L = ½\frac{m_1 m_2 l^2\, \dot\phi^2}{m_1 + m_2} + m_2 \mathrm{g} l(1 - \phi^2/2)$$

As a result, the Lagrange's equation of motion reads

$$\frac{m_1 m_2 l^2}{m_1 + m_2}\,\ddot\phi + m_2\, g l\,\phi = 0$$

or $\ddot\phi + g(m_1 + m_2)/m_1 l\,\phi = 0$

Thus, the frequency of oscillations of the given simple pendulum

$$\omega = [g(m_1 + m_2)/m_1 l]^{½}$$

Note that as $m_1 \to \infty$ (corresponding to fixed support), one returns to the case of the familiar simple pendulum which oscillates with frequency $\omega = (\mathrm{g}/l)^{1/2}$

5.5 From sol. 1.1 (Fig. 1.7)

$$V = V_1 + V_2 = -m_1 g l_1 \cos\phi_1 - m_2 \mathrm{g}(l_1 \cos\phi_1 + l_2 \cos\phi_2)$$
$$= -mgl \cos\phi_1 - mgl\,(\cos\phi_1 + \cos\phi_2),$$

and
$$T = T_1 + T_2 = ½m_1 l_1^2\, \dot\phi_1^2 + ½\,\mathrm{m}_2\,[l_1^2\, \dot\phi_1^2 + l_2^2\dot\phi_2^2$$
$$+ 2\, l_1 l_2\, \dot\phi_1\dot\phi_2 \cos(\phi_1 - \phi_2)]$$
$$= ½\, ml^2\dot\phi_1^2 + ½ml^2[\dot\phi_1^2 + \dot\phi_2^2 + 2\,\dot\phi_1\dot\phi_2 \cos(\phi_1 - \phi_2)]$$
$$= ml^2\dot\phi_1^2 + ½\, ml^2\dot\phi_2^2 + ml^2\ \dot\phi_1\dot\phi_2 \cos\,(\phi_1 - \phi_2)$$

in the present case.

Since ϕ_1 and ϕ_2 are small, we can take

$\cos\phi_1 \to 1 - \phi_1^2,\ \cos\phi_2 \to 1 - \phi_2^2/2$ and $\cos(\phi_1 - \phi_2) \to 1$

As a result

$$V = -mgl(1 - \phi_1^2/2) - mgl(1 - \phi_1^2/2 + 1 - \phi_2^2/2)$$
$$= -3\, mgl + mgl\,\phi_1^2 + ½mgl\,\phi_2^2$$

and
$$T = ml^2\dot\phi_1^2 + ½ml^2\dot\phi_2^2 + ml^2\dot\phi_1\dot\phi_2$$

Lagrangian L of the system would, therefore be given by

$$L = T - V = ml^2\dot{\phi}_1^2 + ½ml^2\dot{\phi}_2^2 + ml^2\,\dot{\phi}_1\dot{\phi}_2 - mg\,l\,\phi_1^2$$
$$- ½\,mgl\,\phi_2^2 + \text{Constt.}$$

As a result, the Lagrange's equations of motion are

$$2\,l\ddot{\phi}_1 + l\ddot{\phi}_2 + 2\,g\phi_1 = 0 \tag{58}$$

and $$l\ddot{\phi}_1 + l\ddot{\phi}_2 + g\phi_2 = 0 \tag{59}$$

Substituting the oscillatory solution,

$$\phi_j = a_j e^{i\omega t},$$

that is, $$\phi_1 = a_1 e^{i\omega t}, \text{ and } \phi_2 = a_2 e^{i\omega t}$$

in equs. (58) and (59), one gets

$$2a_1(g - l\omega^2) - a_2\omega^2 l = 0 \tag{60}$$

$$\text{and } - a_1 l\omega^2 + a_2(g - l\omega^2) = 0 \tag{61}$$

The above equations would have non – vanishing solutions for a_1 and a_2 only if the determinant of their coefficients vanishes, that is

$$\begin{vmatrix} 2(g - l\omega^2) & -l\omega^2 \\ -l\omega^2 & g - l\omega^2 \end{vmatrix} = 0$$

or $$2(g - l\omega^2)^2 - l^2\omega^4 = 0$$

or $$\omega^4 - 4g/l\,\omega^2 + 2\,g^2/l^2 = 0$$

The roots of this equation would give the frequencies of two normal modes, and are given by

$$\omega_{1,2}^2 = \frac{4g/l \pm \sqrt{(4g/l)^2 - 4 \times 2g^2/l^2}}{2}$$

and hence $$\omega_1^2 = (2 + \sqrt{2})g/l \text{ and } \omega_2^2 = (2 - \sqrt{2})g/l$$

Let us now find the coefficients a_j which in the present case can be written in the matrix form as (from equs. (60) and (61))

$$\begin{pmatrix} 2(g/l - \omega^2) & -\omega^2 \\ -\omega^2 & g/l - \omega^2 \end{pmatrix}\begin{pmatrix} a_{1r} \\ a_{2r} \end{pmatrix} = 0$$

where r = 1, 2 correspond to the two frequencies obtained above. Thus

$$2(g/l - \omega^2)\,a_{1r} - \omega^2 a_{2r} = 0$$
$$-\omega^2 a_{1r} + (g/l - \omega^2)a_{2r} = 0$$

For $r = 1$, we have $a_{11} = -1/\sqrt{2}\,a_{21} = -1/\sqrt{2}a_{12}$

For $r = 2$, we have $a_{22} = -\sqrt{2}\,a_{12} = -\sqrt{2}a_{21}$

Therefore, the eigen vectors are

$$\boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = a_{11}\begin{pmatrix} 1 \\ -1/\sqrt{2} \end{pmatrix} \text{ and } \boldsymbol{a}_2 = \begin{pmatrix} a_{22} \\ a_{21} \end{pmatrix} = a_{22}\begin{pmatrix} 1 \\ -1/\sqrt{2} \end{pmatrix}$$

5.7 The equilibrium is taken where both the pendulums are in the vertical position, that is at $x_1 = x_2 = 0$. The displacements x_1 and x_2 to the right are positive while θ_1 and θ_2 are positive in the counter – clockwise direction.

The potential energy of the system measured from their equilibrium position is

$$V = mg\,l(1 - \cos\theta_1) + mg\,l(1 - \cos\theta_2) + \tfrac{1}{2}k(x_1 - x_2)^2$$

where $\tfrac{1}{2}k(x_1 - x_2)^2$ is the potential energy of the spring.

For θ_1 and θ_2 to be small (neglecting terms of the order higher than the quadratic)

$$\begin{aligned} V &= 2\,mgl - mgl(\cos\theta_1 + \cos\theta_2) + \tfrac{1}{2}k(x_1 - x_2)^2 \\ &\approx 2\,mgl - mgl\,(1 - \theta_1^2/2 + 1 - \theta_2^2/2) + \tfrac{1}{2}k(x_1 - x_2)^2 \\ &= \tfrac{1}{2}mgl\,(\theta_1^2 + \theta_2^2) + \tfrac{1}{2}k(x_1 - x_2)^2 \\ &\approx \tfrac{1}{2}mgl\,\{(x_1/l)^2 + (x_2/l)^2\} + \tfrac{1}{2}k(x_1 - x_2)^2 \\ &= \tfrac{1}{2}(k + mg/l)\,x_1^2 + \tfrac{1}{2}(k + mg/l)\,x_2^2 - kx_1x_2 \qquad (62) \end{aligned}$$

Thus the required matrix for the potential energy is

$$V = \begin{bmatrix} k + mg/l & -k \\ -k & k + mg/l \end{bmatrix} \qquad (63)$$

The expression for the kinetic energy is

$$T = \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}\dot{x}_2^2 \qquad (64)$$

Therefore, the required matrix for the kinetic energy is

$$T = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad (65)$$

where the components T_{11}, T_{22} are coefficients of $\tfrac{1}{2}\dot{x}_1^2$ and $\tfrac{1}{2}\dot{x}_2^2$. The components T_{12}, T_{21} are coefficients of $\dot{x}_1\,\dot{x}_2$ and $\dot{x}_2\,\dot{x}_1$.

To determine the normal (or characteristic) frequencies, we use the characteristic (or secular) equ. (22), that is

$$\begin{bmatrix} k + mg/l - m\omega^2 & -k \\ -k & k + mg/l - m\omega^2 \end{bmatrix} = 0$$

or $\quad (k + mg/l - m\omega^2)^2 - k^2 = 0$

or $\quad (k + mg/l - m\omega^2 - k)\,(k + mg/l - m\omega^2 + k) = 0$

Either

$$k + mg/l - m\omega^2 - k = 0$$

which gives $\quad \omega^2 = \omega_1^2 = g/l$ or $\omega_1 = (g/l)^{1/2}$

or $\quad k + mg/l - m\omega^2 + k = 0$

which gives $\quad \omega^2 = \omega_2^2 = g/l + 2k/m$

or $$\omega_2 = (g/l + 2k/m)^{1/2}$$

Let us now proceed to find the coefficients a_j from equ. (20) which in the present case can be written in the matrix form as

$$\begin{pmatrix} g/l + k/m - \omega^2 & -k/m \\ -k/m & g/l + k/m - \omega^2 \end{pmatrix}\begin{pmatrix} a_{1r} \\ a_{2r} \end{pmatrix} = 0$$

where $r = 1, 2$ correspond to the two frequencies obtained above. Thus

$$(g/l + k/m - \omega^2)\, a_{1r} - k/m\, a_{2r} = 0,$$

and $$-k/m\, a_{1r} + (g/l + k/m - \omega^2) a_{2r} = 0$$

For $r = 1$, one gets (substituting $\omega^2 = \omega_1^2 = g/l$ in the 1st equation),

$$a_{11} - a_{21} = 0 \quad \text{or} \quad a_{11} = a_{21} = a_{12}$$

For r = 2, one gets (substituting $\omega^2 = \omega_2^2 = (g/l + 2k/m)$ in the 2nd equation),

$$a_{12} + a_{22} = 0 \quad \text{or} \quad a_{22} = -a_{12} = -a_{21}$$

Therefore, the eigen vectors are

$$\boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = a_{11}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and $$\boldsymbol{a}_2 = \begin{pmatrix} a_{22} \\ a_{21} \end{pmatrix} = a_{22}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

5.8 The system (see Fig. 5.10) in the present ease has three degrees of freedom represented by the coordinates x_1, x_2 and x_3. The configuration of the system is represented by the displacements measured from the equilibrium positions O_1, O_2 and O_3, respectively. The displacements to the right are taken as positive.

The kinetic energy of the system is given by

$$T = \tfrac{1}{2} \times 1 \times \dot{x}_1^2 + \tfrac{1}{2} m\, \dot{x}_2^2 + \tfrac{1}{2} \times 1 \times \dot{x}_3^2$$

Thus the required matrix for the kinetic energy is

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The potential energy of the system measured from the equilibrium position is

$$V = \tfrac{1}{2} \times 1 \times (x_2 - x_1)^2 + \tfrac{1}{2} \times 1 \times (x_3 - x_2)^2$$
$$= \tfrac{1}{2} \times 1 \times (x_1^2 + 2x_2^2 + x_3^2 - 2\, x_1 x_2 - 2\, x_2 x_3)$$

Thus the required matrix for the potential energy is

$$V = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

To determine the characteristic frequencies, we use the secular equ. (22), that is,

$$\begin{bmatrix} 1-\omega^2 & -1 & 0 \\ -1 & 2-\omega^2 m & -1 \\ 0 & -1 & 1-\omega^2 \end{bmatrix} = 0$$

which on solving gives

$$\omega^2 = \omega_1^2 = 0 \text{ (or } \omega_1 = 0), \omega^2 = \omega_2^2 = 1 \text{ (or } \omega_2 = 1),$$

and $$\omega^2 = \omega_3^2 = 1 + 2/m \text{ or } \omega_3 = (1 + 2/m)^{1/2}$$

Let us now try to find the coefficients a_i for each frequency from equ. (20) which in the present case can be written in the matrix form as

$$\begin{pmatrix} 1-\omega^2 & -1 & 0 \\ -1 & 2-\omega^2 m & -1 \\ 0 & -1 & 1-\omega^2 \end{pmatrix} \begin{pmatrix} a_{1r} \\ a_{2r} \\ a_{3r} \end{pmatrix} = 0$$

where $r = 1, 2$ and 3 correspond to the three frequencies obtained above. Thus

$$\begin{aligned} (1-\omega^2)\, a_{1r} - a_{2r} &= 0 \\ -a_{1r} + (2-\omega^2 m)\, a_{2r} - a_{3r} &= 0 \\ -a_{2r} + (1-\omega^2)\, a_{3r} &= 0 \end{aligned} \tag{66}$$

For $r = 1$, one gets (substituting $\omega = \omega_1 = 0$ in the above equation)

$$a_{11} = a_{21} = a_{31}$$

Further using the normalization condition,

$$(a_{1r}^2 + a_{3r}^2) + m\, a_{2r}^2 = 1,$$

one gets $$a_{11} = a_{21} = a_{31} = 1/(2+m)^{1/2}$$

For $r = 2$, using equ. (66) along with the normalization condition, yields

$$a_{22} = 0, a_{12} = -a_{32}$$

and finally $$a_{12} = 1/\sqrt{2}, a_{22} = 0, a_{32} = -1/\sqrt{2}$$

For r = 3, using equ. (66) alongwith the normalization condition yields

$$a_{13} = 1/\{2(1 + 2/m)\}^{1/2},$$

$$a_{23} = -2/\{2m(2+m)\}^{1/2}$$

$$a_{33} = 1/\{2(1 + 2/m)\}^{1/2}$$

5.9 The kinetic energy of the system is given by

$$T = ½ \times 1 \times \dot{x}_1^2 + ½ \times 1 \times \dot{x}_2^2$$

Therefore, the required matrix for the kinetic energy is

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The potential energy of the system is given by

$$V = ½k(x_2 - x_1)^2 + ½ \times 1 \times (x_1 - 0)^2 + ½ \times 1 \times (x_2 - 0)^2$$

$$= ½[(k + 1)\, x_1^2 + (k + 1)\, x_2^2 - 2kx_1 x_2]$$

Therefore, the required matrix for the potential energy is

$$V = \begin{bmatrix} k+1 & -k \\ -k & k+1 \end{bmatrix}$$

To determine the characteristic frequencies, we use the secular equ. (22), that is,

$$\begin{bmatrix} k+1-\omega^2 & -k \\ -k & k+1-\omega^2 \end{bmatrix} = 0$$

or $\quad (k + 1 - \omega^2)^2 - k^2 = 0$

or $\quad (k + 1 - \omega^2) = \pm k$

Therefore,

$$\omega^2 = \omega_1^2 = 1 \text{ (or } \omega_1 = 1) \text{ and } \omega^2 = \omega_2^2 = 2\,k + 1 \text{(or } \omega_2 = (2k + 1)^{1/2})$$

The eigne vector corresponding to $\omega_1^2 = 1$ can be found as follows

$$\begin{pmatrix} k+1-1 & -k \\ -k & k+1-1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore, the normalized eigen vector would be $1/\sqrt{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Similarly for $\omega_2^2 = 2\,k + 1$

$$\begin{pmatrix} k+1-(2k+1) & -k \\ -k & k+1-(2k+1) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and hence the normalized eigen vector is $1/\sqrt{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

5.10 Let x_1, x_2 and x_3 represent the displacements of the atoms from the equilibrium positions (see Fig. 5.12). As a result, the kinetic and potential energies of the system would be given by

$$T = ½m\dot{x}_1^2 + ½M\dot{x}_2^2 + ½m\dot{x}_3^2$$

and

$$V = ½\, k(x_2 - x_1)^2 + ½\, k(x_3 - x_2)^2$$

$$= ½\, k(x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3)$$

Therefore, the required matrices for the kinetic and potential energies would be given by

$$T = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \qquad (67)$$

and
$$V = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \qquad (68)$$

To determine the characteristic frequencies, we use the secular equ. (22), that is,

$$\begin{bmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{bmatrix} = 0$$

which amounts to

$$(k-\omega^2 m)\,[(2k-\omega^2 M)\,(k-\omega^2 m)-k^2] + k[-k(k-\omega^2 m)] = 0$$

or $(k-\omega^2 m)^2\,(2k-\omega^2 M) - k^2\,(k-\omega^2 m) - k^2\,(k-\omega^2 m)] = 0$

or $(k-\omega^2 m)\,\omega^2\,\{-k(M+2m)+\omega^2 Mm\} = 0$ (69)

which gives the following solutions for the characteristic frequencies

$\omega_1 = 0$, $\omega_2 = (k/m)^{1/2}$, and $\omega_3 = \{k/m\,(1+2m/M)\}^{1/2}$ (70)

Since the zero frequency found above is of no consequence for the vibrational frequencies of interest, one can eliminate this root by imposing the condition that the centre of mass of the molecule remains stationary at the origin,

$$m(x_1+x_3)+Mx_2 = 0 \qquad (71)$$

The above equation can then be used to eliminate one of the coordinates from T and V and thereby reducing the problem to one of two degrees of freedom instead of three.

Regarding ω_2, one can infer that it would be recognized as the well known frequency of oscillations for a mass m suspended by a spring of strength k. We are, therefore, lead to the conclusions that only the end atoms take part in this vibration, the centre of the molecule remains stationary. It is only in the third mode of vibration, ω_3, that the mass M can participate in the oscillatory motion. These predictions are truly speaking verified by examining the eigen vectors for the three modes obtained above.

Let us now try to find the coefficients a_j for each frequency from equ. (20) which in the present case can be written in the matrix form as

$$\begin{pmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{pmatrix} \begin{pmatrix} a_{1r} \\ a_{2r} \\ a_{3r} \end{pmatrix} = 0$$

where r = 1, 2, and 3 correspond to the three frequencies determined above. Thus

$$
\begin{aligned}
&(k - \omega^2 m)\, a_{1r} - k\, a_{2r} = 0 \\
&- k a_{1r} + (2k - \omega^2 M) a_{2r} - k a_{3r} = 0 \\
&- k a_{2r} + (k - \omega^2 m) a_{3r} = 0
\end{aligned}
\tag{72}
$$

along with the normalization condition.

$$m(a_{1r}^2 + a_{3r}^2) + M\, a_{2r}^2 = 1 \tag{73}$$

For $r = 1$, from first and third of equs. (72), after substituting $\omega = \omega_1 = 0$, one gets

$$a_{11} = a_{21} = a_{31}$$

Using the normalization condition (73), one gets

$$a_{11} = 1/(2m + M)^{½}, a_{21} = 1/(2\, m + M)^{½}, a_{31} = 1/(2m + M)^{½} \tag{74a}$$

that is, all the three coefficients are equal. This, of course, is exactly what was expected from the translational nature of the motion (see Figs. 5.14a).

For $r = 2$, the factors $(k - \omega^2 m)$ vanish for the second mode $(\omega_2^2 = k/m)$, and equs. (72) give $a_{22} = 0$ (as predicted) and $a_{12} = - a_{32}$. Using the normalization condition (73), one gets

$$a_{12} = 1/(2m)^{½},\ a_{22} = 0,\ a_{32} = 1/(2m)^{½} \tag{74b}$$

In this mode the central atom is at rest, while the two outer atoms vibrate exactly out of phase (see Fig. 5.14b).

Finally for $r = 3$, for the third mode $(\omega_3^2 = k/m + 2k/M)$, from first of equs. (72),

$$a_{13} = -\frac{M}{2m}\, a_{23} \tag{75}$$

and from second of equs. (72),

$$-a_{13} - \frac{M}{m} a_{23} - a_{33} = 0$$

Substituting the value of a_{23} from equ. (75) into above, one gets

$$a_{13} = a_{33} \tag{76}$$

Further from the normalization condition

$$m\left(a_{13}^2 + a_{33}^2\right) + M\, a_{23}^2 = 1,$$

using equs. (75) and (76), one gets

$$2m\, a_{13}^2 + M \times (-2m/M)\, a_{13}^2 = 1$$

and finally

$$
\begin{aligned}
&a_{13} = 1/\sqrt{2m(1 + 2m/M)},\quad a_{23} = -2/\sqrt{2M(2 + M/m)}, \\
&a_{33} = 1/\sqrt{2m(1 + 2m/M)}
\end{aligned}
\tag{74c}
$$

Therefore, in the 3rd mode the two outer atoms vibrate with the same amplitude, while the central one oscillates out of phase with them and has a different amplitude (see Fig. 5.14c).

5.11 The Lagrangian of the system is given by

$$L = ½m(\dot{x}_1^2 + \dot{x}_3^2) + ½M\dot{x}_2^2 - ½k[(x_1 - x_2)^2 + (x_3 - x_2)^2]$$

Let us assume that the centre of mass of the system remains at rest. That is,

$$m(x_1 + x_3) + Mx_2 = 0$$

Using above, and eliminating x_2 from the Lagrangian L, we get

$$L = \mu m/4M\,\dot{X}_1^2 + m/4\,\dot{X}_2^2 - k\mu^2/4M^2\,X_1^2 - k/4\,X_2^2$$

where $\mu = 2m + M$ is the mass of the molecule and the new coordinates

$$X_1 = x_1 + x_3, X_2 = x_1 - x_3.$$

The Lagrange's equations of motion are, therefore, given by

$$\frac{\mu m}{2M}\ddot{X}_1 + \frac{k\mu^2}{2M^2}X_1 = 0 \text{ or } \ddot{X}_1 + \frac{k\mu}{mM}X_1 = 0$$

and $$\frac{m}{2}\ddot{X}_2 + \frac{k}{2}X_2 = 0 \text{ or } \ddot{X}_2 + \frac{k}{m}X_2 = 0$$

Thus X_1 and X_2 are the normal coordinates of the system and corresponding frequencies are $\omega_1 = (k\mu/mM)^{1/2}$ and $\omega_2 = (k/m)^{1/2}$, respectively.

If mode X_1 is excited, then X_2 must be suppressed, that is,

$$X_2 = x_1 - x_3 = 0 \text{ or } x_1 = x_3$$

which corresponds to a vibration antisymmetrical about the centre of mass of the molecule (see Fig. 5.12*a*). Similarly if mode X_2 is excited, then X_1 must be suppressed, that is,

$$X_1 = x_1 + x_3 = 0 \text{ or } x_1 = -x_3$$

which corresponds to a symmetrical vibration about the centre of mass of the molecule (see Fig. 5.12*b*).

5.12 The Lagrange's equations of motion in the present case are

$$\ddot{x} + \omega_0^2 x = \alpha y, \tag{77}$$

and $$\ddot{y} + \omega_0^2 y = \alpha x$$

Substituting the oscillatory solutions,

$$x = Ae^{i\omega t} \tag{78}$$

and $$y = Be^{i\omega t}$$

in the above equations, one gets

$$A(\omega_0^2 - \omega^2) = \alpha B,$$

and $$B(\omega_0^2 - \omega^2) = \alpha A \tag{79}$$

The characteristic equation, therefore, reads

$$(\omega_0^2 - \omega^2) - \alpha^2 = 0,$$

which on solving gives,

$$\omega^2 = \omega_1^2 = \omega_0^2 - \alpha,$$

and $$\omega^2 = \omega_2^2 = \omega_0^2 + \alpha$$

For $\omega = \omega_1$, equ. (79) given

$$A/B = 1 \tag{80a}$$

and for $\omega = \omega_2$, it gives

$$A/B = -1 \tag{80b}$$

As a result, the general solutions of equs. (77) may be written as

$$\begin{aligned} x &= A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t} + A_2 e^{i\omega_2 t} + A_{-2} e^{-i\omega_2 t} \\ y &= B_1 e^{i\omega_1 t} + B_{-1} e^{-i\omega_1 t} + B_2 e^{i\omega_2 t} + B_{-2} e^{-i\omega_2 t} \end{aligned} \tag{81}$$

Combining equs. (81) with equs. (80), we obtain

$$\begin{aligned} x &= A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t} + A_2 e^{i\omega_2 t} + A_{-2} e^{-i\omega_2 t} \\ y &= A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t} - \mathrm{A}_2 e^{i\omega_2 t} - A_{-2} e^{-i\omega_2 t} \end{aligned} \tag{82}$$

where A_1, A_{-1}, A_2, A_{-2} are the constants. After determining the constants from the initial conditions, one notices that the each coordinate (x or y)depends on two frequencies ω_1 and ω_2. Let us now try to find the new coordinates X and Y such that each coordinate oscillates with a single frequency.

From equs. (82),

$$X = x + y = Ce^{i\omega_1 t} + De^{-i\omega_1 t} \tag{83}$$

$$Y = x - y = Ee^{i\omega_2 t} + Fe^{-i\omega_2 t}$$

where C, D, E, and F are new constants.

The coordinates X and Y correspond to the modes of oscillation involving only one frequency ω_1 or ω_2. These are called normal modes and the corresponding coordinates are called the normal coordinates.

Let us now try to obtain the equations of motion in terms of the normal coordinates. To do so, we find from equs. (83),

$$x = \frac{X + Y}{2} \tag{84}$$

and $$y = \frac{X - Y}{2}$$

Substituting the values of x and y from the above equation into equs. (77), we get

$$\frac{\ddot{X} + \ddot{Y}}{2} + \omega_0^2 \left(\frac{X + Y}{2}\right) = \alpha \left(\frac{X - Y}{2}\right),$$

and $\frac{\ddot{X}-\ddot{Y}}{2}+\omega_0^2\left(\frac{X-Y}{2}\right)=\alpha\left(\frac{X+Y}{2}\right)$

Adding and subtracting the above equations, we get

$$\ddot{X}+(\omega_0^2-\alpha)X=0, \tag{85}$$

and $$\ddot{Y}+(\omega_0^2+\alpha)Y=0$$

5.13 Let x_1 and x_2 be the positions of masses m_1 and m_2 measured from the equilibrium positions O_1 and O_2 (see Fig. 5.13).

The kinetic and potential energies of the system are, therefore

$$T=½m_1\dot{x}_1^2+½\,m_2\dot{x}_2^2$$

and $$V=½k(x_2-x_1)^2 \tag{86}$$

It is possible to find new coordinates X_1 and X_2 which are linear combinations of x_1 and x_2, such that each new coordinate oscillates with a single frequency. These coordinates are termed as the normal coordinates and are given by

$$X_1=x_1+x_2$$

and $$X_2=x_1-x_2 \tag{87}$$

From the above equations

$$x_1=\frac{X_1+X_2}{2},$$

and $$x_2=\frac{X_1-X_2}{2} \tag{88}$$

The Lagrangian of the system is given by

$$L=T-V=½\,m_1\dot{x}_1^2+½m_2\dot{x}_2^2-½k(x_2-x_1)^2$$

and hence the Lagrange's equations of motion are

$$m_1\ddot{x}_1-k(x_2-x_1)=0,$$

and $$m_2\ddot{x}_2+k(x_2-x_1)=0 \tag{89}$$

Substituting the values of x_1 and x_2 from equs. (88) into above and then adding and subtracting the resulting equations, we get

$$\ddot{X}_1+0\,X_1=0, \tag{90}$$

and $\ddot{X}_2+2k/(m_1+m_2)\,X_2=0$

Thus the frequencies of oscillations are

$$\omega=\omega_1=0, \quad \text{and} \quad \omega=\omega_2=(2k/(m_1+m_2)]^{½} \tag{91}$$

If mode X_1 is excited, then X_2 must be suppressed, that is,

$$X_2=x_1-x_2=0 \quad \text{or} \quad x_1=x_2$$

which corresponds to uniform translational motion of the system as shown in Fig. 5.15(*a*). Similarly, if mode X_2 is excited, then X_1 must be suppressed, that is,

$$X_1 = x_1 + x_2 = 0 \quad \text{or} \quad x_1 = -x_2$$

which indicates that the two masses oscillate relative to the centre of mass as shown in Fig. 5.15(b).

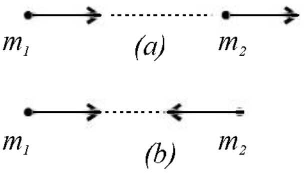


Fig. 5.15 Two possible normal modes of vibration of the system of Fig. 5.13 (*a*) corresponds to uniform translation ($\omega = \omega_1 = 0$). (b) Corresponds to oscillations with frequency, $\omega = \omega_2 = [2k/(m_1 + m_2)]^{1/2}$

5.14 The kinetic energy of the system (see Fig. 5.16)

$$T = 1/2\, m\dot{x}^2 + 1/2m\,(\omega x \sin 30°)^2$$

$$= 1/2\, m\dot{x}^2 + 1/8 m\omega^2 x^2$$

and the potential energy

$$V = 1/2k(x - l)^2 - mg\,x\cos 30°$$

$$= 1/2k(x - l)^2 - mg\,x\sqrt{3}/2$$

As a result, the Lagrangian L would be given by

$$L = T - V = \tfrac{1}{2}\, m\dot{x}^2 + 1/8 m\omega^2 x^2 - \tfrac{1}{2}k(x - l)^2 + mg\,x\sqrt{3}/2$$

and hence the Lagrange's equation of motion reads

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

or

$$m\ddot{x} = \tfrac{1}{4}\, m\omega^2 x - k(x - l) + mg\sqrt{3}/2 \qquad (93)$$

Let $x = x_0$ be the equilibrium point, therefore

$$0 = \tfrac{1}{4}\, m\omega^2 x_0 - k(x_0 - l) + mg\sqrt{3}/2 \qquad (94)$$

To discuss the oscillations about the equilibrium point, let us put

$$\xi = x - x_0 \qquad (95)$$

Subtracting equ. (94) from equ. (93), one gets

$$m\ddot{\xi} + (k - \tfrac{1}{4}\, m\omega^2)\,\xi = 0$$

or

$$\ddot{\xi} + (k/m - \tfrac{1}{4}\omega^2)\,\xi = 0$$

Therefore, the frequency of oscillations is given by

$$\omega'^2 = k/m - \tfrac{1}{4}\,\omega^2$$

5.15 Let y_1 and y_2 be the displacements of the masses M and m from their equilibrium positions (See Fig. 5.17). As a result, the kinetic and potential energies of the system would be given by

$$T = ½M\dot{y}_1^2 + ½m\dot{y}_2^2$$

and
$$V = ½\,Ky_1^2 + ½k(y_2 - y_1)^2 - Mg\,y_1 - mg\,y_2$$

and hence the Lagrangian

$$L = T - V = ½M\dot{y}_1^2 + ½\,m\dot{y}_2^2 - ½\,Ky_1^2 - ½k(y_2 - y_1)^2 + Mg\,y_1 + mg\,y_2$$

The Lagrange's equations of motion are, therefore, given by

$$\ddot{y}_1 + K/M\,y_1 - k/M\,(y_2 - y_1) - g = 0, \tag{96}$$

and
$$\ddot{y}_2 + k/m\,(\mathrm{y}_2 - \mathrm{y}_1) - \mathrm{g} = 0 \tag{97}$$

which are coupled differential equations in the variables y_1 and y_2.

To apply the normal mode theory, the potential energy must be a quadratic function of coordinates. This would be possible only if the linear gravitational terms in V could some how be eliminated. To achieve the required outcome, let us make a linear transformation to new coordinates X_1 and X_2 such that

$$X_1 = y_1 - \frac{(M+m)g}{K},$$

$$X_2 = y_2 - \frac{(M+m)g}{K} - \frac{mg}{k} \tag{98}$$

In terms of the above coordinates, the potential energy becomes

$$V = ½K\left[X_1 + \frac{(M+m)g}{K}\right]^2 + ½k\left[\left\{X_2 + \frac{(M+m)g}{K} + \frac{mg}{k}\right\} - \left\{X_1 + \frac{(M+m)g}{K}\right\}\right]^2 - Mg\left[X_1 + \frac{(M+m)g}{K}\right] - mg\left[X_2 + \frac{(M+m)g}{K} + \frac{mg}{k}\right]$$

$$= ½\,KX_1^2 + ½k(X_2 - X_1)^2 + \text{constant}$$

That is, the transformation (98) was so designed as to eliminate the linear terms in V. The constant term is, of course, irrelevant so far as the Lagrangian derivatives are concerned and hence can be omitted. The kinetic energy reads as

$$T = ½M\dot{X}_1^2 + ½m\dot{X}_2^2$$

Therefore, the required matrices for the kinetic and potential energies would be given by

$$T = \begin{bmatrix} M & O \\ O & m \end{bmatrix}, \quad V = \begin{bmatrix} K+k & -k \\ -k & k \end{bmatrix}$$

To determine the characteristic frequencies, we use the secular equ. (22) which results in

$$\begin{bmatrix} K+k-\omega^2 M & -k \\ -k & k-\omega^2 m \end{bmatrix} = 0$$

or $\quad (K + k - \omega^2 M)\,(k - \omega^2 m) - k^2 = 0$

or $\quad \omega^4\, Mm - \omega^2\,\{kM + (K + k)m\} + Kk = 0$

Therefore, the two values of frequency squared are given by

$$\omega^2 = \frac{kM + (K+k)m \pm \sqrt{\{kM + (K+k)m\}^2 - 4MmKk}}{2Mm} \tag{99}$$

The Lagrangian in the new coordinates is

$$L = T - V = ½M\dot{X}_1^2 + ½m\dot{X}_2^2 - ½KX_1^2 - ½\,k(X_2 - X_1)^2$$

and as a result, the modified Lagrange's equations of motion are

$$\ddot{X}_1 + \frac{K}{M}X_1 - \frac{k}{M}(X_2 - X_1) = 0, \tag{96$'$}$$

$$\ddot{X}_2 + \frac{k}{m}(X_2 - X_1) = 0, \tag{97$'$}$$

Multiplying (96′) by a constant A and (97′) by a constant *B* and then adding, the resulting equation reads

$$A\ddot{X}_1 + B\ddot{X}_2 + \left[\frac{A}{M}(K+k) - \frac{Bk}{m}\right]X_1 + \left[\frac{B}{m} - \frac{A}{M}\right]kX_2 = 0 \tag{100}$$

Normal coordinates being linear combinations of the terms

$$Q = AX_1 + BX_2$$

and the Lagrange's equations being of simple harmonic oscillator type in these coordinates, we get

$$\ddot{Q} + \omega^2 Q = 0 = A\;\ddot{\ }\; + B\ddot{X}_2 + \omega^2(AX_1 + BX_2) \tag{101}$$

Identifying the multiplication factors above with the *A* and *B* of the linear combination for *Q*, the combined equation of motion (100) has exactly the above form provided we identify coefficients of the linear terms

$$\omega^2 A = A/M(K + k) - B\,k/m,\ \omega^2 B = (B/m - A/M)k$$

Dividing one relation by the other and on simplification, we get

$$(A/B)^2 + \left(\frac{K+k}{k} - \frac{M}{m}\right) A/B - M/m = 0$$

For the two values of A/B, one gets two normal coordinates as

$$Q = B(A/B\, X_1 + X_2) = B\left[A/B\, y_1 + y_2 - \frac{(M+m)g}{K}\left(\frac{A}{B}+1\right) - \frac{mg}{k}\right]$$

5.16 Taking origin at the fixed point 0 of the ring, the position of the bead is given by (see Fig. 5.18).

$$x = a\sin\theta + a\sin\phi,$$

$$y = -a\cos\theta - a\cos\phi$$

Therefore, the kinetic energy of the bead

$$T_{\text{bead}} = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$= \tfrac{1}{2} ma^2\left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos\left(\phi - \theta\right)\right]$$

Further, according to the parallel axis theorem, the moment of inertia of the ring about the point of support is $2Ma^2$, therefore, the kinetic energy of the ring would be given by

$$T_{ring} = \tfrac{1}{2} I\dot{\theta}^2 = M a^2 \dot{\theta}^2$$

The total kinetic energy of the system as a result of the above expressions would be

$$T = T_{\text{bead}} + T_{\text{ring}}$$

$$= \tfrac{1}{2} a^2[(2M + m)\dot{\theta}^2 + m\dot{\phi}^2 + 2\, m\dot{\theta}\,\dot{\phi}]$$

where we have taken $\cos(\phi - \theta) \approx 1$ under the small oscillations approximation to retain the quadratic form for the kinetic energy T with constant coefficients.

The potential energy as measured from the equilibrium position is

$$V = Mg\, a(1 - \cos\theta) + mg\, a(2 - \cos\theta - \cos\phi)$$

$$= g\, a/2\ [(M + m)\,\theta^2 + m\,\phi^2]$$

where for small angles, we have taken $1 - \cos\theta \approx 1 - (1 - \theta^2/2) = \theta^2/2$ and similarly $1 - \cos\phi \approx \phi^2/2$. As a result, the kinetic and potential energies matrices would be given by

$$T = \begin{bmatrix} (2M + m)a & ma \\ ma & ma \end{bmatrix}$$

and

$$V = \begin{bmatrix} (M + m)g & 0 \\ 0 & mg \end{bmatrix}$$

To determine the allowed (characteristic) frequencies, we use the secular equ. (22), that is

$$\begin{vmatrix} (M + m)g - \omega^2(2M + m)a & -\omega^2 ma \\ -\omega^2 ma & mg - \omega^2 ma \end{vmatrix} = 0$$

or $$\{(M + m)g - \omega^2(2M + m)a\}\ (mg - \omega^2 ma) - \omega^4\, m^2\, a^2 = 0$$

On simplification, the roots are given by

$$\omega_1^2 = g/2a \ \text{ and } \ \omega_2^2 = (m + M)\, g/Ma$$

Let us now find the coefficients a_j which in the present case can be written in the matrix form as

$$\begin{pmatrix} (M+m)g - \omega^2(2M+m)a & -\omega^2 ma \\ -\omega^2 ma & mg - \omega^2 ma \end{pmatrix} \begin{pmatrix} a_{1r} \\ a_{2r} \end{pmatrix} = 0$$

where $r = 1, 2$ correspond to the two frequencies found above. Thus

$$[(M + m)g - \omega^2\,(2M + m)\,a]\ a_{1r} - \omega^2\, ma\, a_{2r} = 0,$$

$$-\,\omega^2\, ma\, \mathrm{a}_{1\mathrm{r}} + (mg - \omega^2 ma)\ a_{2r} = 0$$

For $r = 1$, we have

$$a_{11} = a_{21}$$

For $r = 2$, we have

$$a_{22} = -\frac{M+m}{m}\, a_{12}$$

Therefore, the eigen vectors are

$$\boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = a_{11} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{a}_2 = \begin{pmatrix} a_{22} \\ a_{21} \end{pmatrix} = a_{22} \begin{pmatrix} 1 \\ \dfrac{-m}{M+m} \end{pmatrix}$$

6

CHAPTER

Hamilton's Equations of Motion

6.1 INTRODUCTION

So far we have developed the formulation of the laws of mechanics in terms of the Lagrangian and the Lagrange's equations of motion derived from it. In doing so it was assumed that the mechanical state of the system is described by specifying its generalized coordinates and generalized velocities. However, this is not only the possible mode of description. There exists another mode of description known as Hamiltonian formulation. In this approach the state of the system is described in terms of generalized coordinates and generalized momenta. This is truly speaking an alternate but more powerful and convenient method of working with the physical principles already established. The equations of motion obtained are first-order linear differential equations instead of second order in the Lagrangian formulation. The formulation forms the basis for the development of Hamilton-Jacobi theory.

6.2 HAMILTON'S EQUATIONS OF MOTION

The Lagrangian L is a function of generalized coordinates and generalized velocities and may also be an explicit function of time, that is,

$$L = L(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots\ldots\ldots, \dot{q}_n; t) \qquad (1)$$

The total differential of L is

$$dL = \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial L}{\partial t} dt$$

Since the generalized momentum, $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ by definition, therefore, from the Lagrange's equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i}, \text{ we obtain from equ. (1),}$$

$$dL = \sum_i \left(\dot{p}_i dq_i + p_i d\dot{q}_i\right) + \frac{\partial L}{\partial t} dt$$

Adding $\dot{q}_i \, dp_i$ on both sides of the above equation, after rearranging, we get

$$d(\sum_i p_i \dot{q}_i - L) = \sum_i (\dot{q}_i \, dp_i - \dot{p}_i \, dq_i) - \partial L/\partial t \, dt \tag{2}$$

As before, we define the Hamilton function H also called Hamiltonian of the system to be

$$H = \sum_i p_i \dot{q}_i - L(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t) \tag{3}$$

As a result equ. (2) takes the form

$$dH = \sum_i (\dot{q}_i \, dp_i - \dot{p}_i \, dq_i) - \partial L/\partial t \, dt \tag{4}$$

We want to express H as a function of generalized coordinates and generalized momenta (also as function of time). This can be done by using the relation, $\partial L/\partial \dot{q}_i = p_i$, that is, we can express $\dot{q}_i$ in terms of p_i. By doing so, we can write

$$H = H(q_1, q_2, \ldots, q_n; p_1, p_2, \ldots, p_n; t) \tag{5}$$

Thus, using equ. (4), we can write the total differential of H as

$$dH = \sum_i (\partial H/\partial q_i \, dq_i + \partial H/\partial p_i \, dp_i) + \partial H/\partial t \, dt \tag{6}$$

Comparing equs. (6) and (4), we obtain

$$\dot{q}_i = \partial H/\partial p_i \tag{7}$$

$$-\dot{p}_i = \partial H/\partial q_i \tag{8}$$

and

$$-\partial L/\partial t = \partial H/\partial t \tag{9}$$

Equs. (7) and (8) are called Hamilton's equations of motion. They form a set of 2n first order differential equations replacing the n second order differential equations in the Lagrangian formulation. Because of their symmetry of form, they are also called canonical equations of motion.

6.3 CYCLIC COORDINATES AND CONSERVATION THEOREMS

We know that a cyclic coordinate q_i is the one that does not appear explicitly in the Lagrangian. Therefore, by virtue of Lagrange's equations, its conjugate momentum p_i is a constant. This simply implies that

$$\dot{p}_i = 0 = -\partial H/\partial q_i$$

or

$$\partial H/\partial q_i = 0,$$

that is, H does not contain q_i. That means a coordinate that is cyclic would also be absent from the Hamiltonian. Conversely if a generalized coordinate does not appear in H, the conjugate momentum is conserved. The momentum conservation theorems of sect. 2.5 can thus be transferred to the Hamiltonian formulation with nothing more than substitution of H for L.

We have already shown in sec. 2.5 that if L(and hence in consequence of equ. (9) also H) is not an explicit function of t, then H is a constant of motion. This can also be seen directly from the equations of motion (7) and (8) by writing the total time derivative of the Hamiltonian $H = H(q_1, q_2, \ldots, q_n; p_1, p_2, \ldots p_n; t)$ as

$$\frac{dH}{dt} = \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}$$

In consequence of the equations of motion (7) and (8), the first two sums on the r.h.s. cancel each other and therefore, one gets

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

and hence from equ. (9)

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{10}$$

Thus if t does not appear explicitly in L, it would also not be present in H, and H would be constant in time or would be constant of motion.

Further, it was proved in sec. 2.5 that if transformation equations that define the generalized coordinates,

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n; t)$$

do not depend explicitly on t, and if the potential energy is velocity independent then H is the total energy $T + V$ of the system. The identification of H as a constant of motion and as the total energy are two separate matters and the conditions sufficient for one are not enough for the other. It may happen that the transformation equations mentioned above do involve t explicitly but that H does not. In that case H is a constant of motion but it is not the total energy.

6.4 DERIVATION OF HAMILTON'S EQUATIONS OF MOTION FROM VARIATIONAL PRINCIPLE

From variational (or Hamilton's) principle, we have

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \tag{11}$$

As we know, the Hamiltonian function is given by

$$\mathrm{H} = \sum_i p_i \dot{q}_i - L \tag{12}$$

The equations of motion associated with the Hamiltonian $H(q, p, t)$ can be derived easily by rewriting Hamilton's principle, using equ. (12),

$$\delta \int_{t_1}^{t_2} \left[\sum_i p_i \dot{q}_i - H(q, p, t)\right] dt = 0 \tag{13}$$

The above equation is sometimes referred to as the modified Hamilton's principle. Though it would be used most frequently in connection with the transformation theory (to be discussed in the next chapter), but here we would show that the principle leads to Hamilton's equations of motion.

Carrying out the variation, we find from equ. (13),

$$\int_{t_1}^{t_2}\left[\sum_i(p_i\,\delta\dot{q}_i+\dot{q}_i\,\delta p_i)-\sum_i\left(\frac{\partial H}{\partial q_i}\delta q_i+\frac{\partial H}{\partial p_i}\delta p_i\right)\right]dt=0 \qquad (14)$$

Note that we have not considered the variation with time, that is $\delta t = 0$ because of the fact that the time of travel along every varied path is same. Truly speaking the δ variation corresponds to the virtual displacements in which the time is held fixed and only the coordinates vary.

Further, since d and δ variations can be interchanged as the later is, at constant time, the first term in equ. (14) becomes

$$\int_{t_1}^{t_2}\sum_i p_i\,\delta\dot{q}_i\,dt=\int_{t_1}^{t_2}\sum_i p_i\frac{d}{dt}(\delta q_i)dt$$

Integrating the *r.h.s.* by parts, we get

$$\int_{t_1}^{t_2}\sum_i p_i\,\delta\dot{q}_i\,dt=\sum_i p_i\,\delta q_i\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\sum_i\dot{p}_i\,\delta q_i\,dt=-\int_{t_1}^{t_2}\sum_i\dot{p}_i\,\delta q_i\,dt \qquad (15)$$

as there are no variations at the ends or $\delta q_i(t_1)=\delta q_i(t_2)=0$.

Using equ. (15), equ. (14) becomes

$$\int_{t_1}^{t_2}\left[\sum_i\left(-\dot{p}_i-\frac{\partial H}{\partial q_i}\right)\delta q_i+\left(\dot{q}_i-\frac{\partial H}{\partial p_i}\right)\delta p_i\right]dt=0$$

Since in the Hamiltonian formulation q_i and p_i are independent variables, their variations δq_i and δp_i would also be independent of each other. Hence the above integral would vanish only if their coefficients vanish separately. Thus, we get

$$\dot{q}_i=\frac{\partial H}{\partial p_i}, \qquad (16a)$$

$$-\dot{p}_i=\frac{\partial H}{\partial p_i} \qquad (16b)$$

These are Hamilton's equations of motion.

6.5 THE PRINCIPLE OF LEAST ACTION

This is another variational principle associated with the Hamiltonian formulation and was discovered in 1745 by Pierre-Louis Moreau de Maupertuis. It involves a new type of variation, which we would call the Δ-variation. In the δ-variation process used in the discussion of Hamilton's principle, the system must travel from a point A in the configuration space at time t_1 to another point B at time t_2 irrespective of the path, that is whether it is the correct path or the varied path. In other words, the end points t_1 and t_2 are fixed. To obtain Lagrange's equations of motion, we also required that the varied paths return to the same end points in the configuration space, that is, $\delta q_i(t_1) = \delta q_i(t_2) = 0$ or in other words there are no variations at the end points. On the other hand in the Δ-variation, we assume that the Lagrangian and therefore, the Hamiltonian do not involve time explicitly, so that energy of the system is conserved, that is, $H(q, p) = E =$ constant along any path (actual or varied). Also in general the varied path over which an integral is evaluated may end at different times than the correct path, and there may be variation in the coordinates at the end points. That is, the Δ-variation includes the variation of both space and time variables.

Summarizing, we can state that the δ-variation satisfies the following conditions:

(i) δq_i are arbitrary except for the end points, where $\delta q_i = 0$

(ii) $\delta t = 0$

whereas the new variation Δ which includes variations in both space and time, and is defined as

$$\Delta = \delta + \Delta t \, d/dt \tag{17}$$

with the conditions

(i) Δq_i are arbitrary, except for the end points, where $\Delta q_i = 0$,

(ii) Δt is arbitrary

Applying Δ-variation to q_i, we get

$$\Delta q_i = \delta q_i + \Delta t \, \dot{q}_i \tag{18}$$

which expresses the correspondence between two points, one (P) on the correct and the other (P') on the varied path (see Fig. 6.1). At the end points $\delta q_i \neq 0$, but

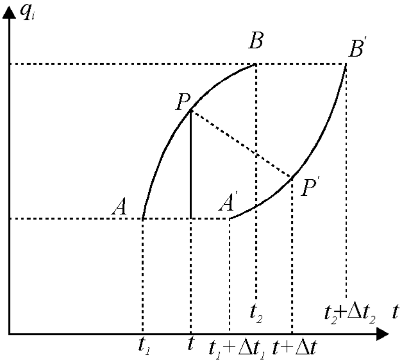


Fig. 6.1 Correspondence between two points, one on the correct (AB) and the other on the varied ($A'B'$) path when both space and time variations are considered.

$$\Delta q_i = 0 \tag{19}$$

Let us now apply the Δ-variation to some function $f = f(q, \dot{q}, t)$ which gives

$$\Delta f = \delta f + \frac{df}{dt}\Delta t$$

$$= \sum_i \left(\frac{\partial f}{\partial q_i}\delta q_i + \frac{\partial f}{\partial \dot{q}_i}\delta \dot{q}_i \right) + \sum_i \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial \dot{q}_i}\ddot{q}_i \right)\Delta t + \frac{\partial f}{\partial t}\Delta t$$

as $$\frac{\partial f}{\partial t}\delta t = 0$$

Using equ. (18), we get

$$\Delta f = \sum_i \left(\frac{\partial f}{\partial q_i}\Delta q_i + \frac{\partial f}{\partial \dot{q}_i}\Delta \dot{q}_i \right) + \frac{\partial f}{\partial t}\Delta t \tag{20}$$

which is the usual differential of $f(q, \dot{q}, t)$

Further, we apply Δ-variation to the Action integral I which amounts to

$$\Delta I = \Delta \int_{t_1}^{t_2} L\,dt = \Delta A|_{t_1}^{t_2} = \Delta A\,(t_2) - \Delta A(t_1) \tag{21}$$

in which $A(t)$ represents the primitive function of Lagrangian L.

Using equ. (17), we get

$$\Delta I = [\delta A(t_2) + \dot{A}\,(t_2)\Delta t_2] - [\delta A(t_1) + \dot{A}\,(\mathrm{t}_1)\,\Delta \mathrm{t}_1]$$

$$= \delta A(t_2) - \delta A(t_1) + \dot{A}\,(t_2)\,\Delta t_2 - \dot{A}\,(t_1)\Delta t_1$$

$$= \delta \int_{t_1}^{t_2} L\,dt + L\ \Delta t|_{t_1}^{t_2} \tag{22}$$

We know that

$$\delta L = \sum_i \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right)$$

$$= \sum_i (\dot{p}_i \delta q_i + p_i \delta\,\dot{q}_i)$$

$$= \frac{d}{dt}(\sum_i p_i\,\delta q_i) = \frac{d}{dt}\sum_i (p_i\,\Delta q_i - p_i \dot{q}_i\,\Delta t) \tag{23}$$

where use of equ. (18) has been made.

Putting equ. (23) into equ. (22) and using $\Delta q_i = 0$ at the end points, we get

$$\Delta I = \Delta \int_{t_1}^{t_2} L\,dt = (\sum_i\ - p_i\ \dot{q}_i\ + L)\ \Delta t|_{t_1}^{t_2}$$

$$= -H\ \Delta t|_{t_1}^{t_2} \tag{24}$$

where use has been made of the relation, $H = \sum_i\ p_i\ \dot{q}_i\ - L$

Since the system is conservative, therefore

$$H = E = T + V = \text{Const.},$$

which means on any varied path $A'B'$, the energy has the same value as on the correct path. Thus

$$H\,\Delta t\Big|_{t_1}^{t_2} = H(\Delta t_2 - \Delta t_1) = H\Delta\int_{t_1}^{t_2} dt = \Delta\int_{t_1}^{t_2} H\,dt \qquad (26)$$

Introducing the above result into equ. (24) and further using the relation $H = \sum\limits_i p_i\,\dot{q}_i - L$, we get

$$\Delta\int_{t_1}^{t_2} \sum_i p_i\,\dot{q}_i\,dt = 0 \qquad (27)$$

which is one of the form of the principle of least action and the quantity

$$W = \int_{t_1}^{t_2} \sum_i p_i\,\dot{q}_i\,dt \qquad (28)$$

is called Maupertuision action.

The principle of least action can be exhibited in a variety of forms. For example, if the transformation equations do not contain time explicitly, the kinetic energy T is a quadratic homogeneous function of generalized velocities $\dot{q}_i$'s and in addition, if the potential energy is not velocity dependent the conjugate momenta are derived from T only, and in consequence

$$\sum_i p_i\dot{q}_i = \sum_i \frac{\partial T}{\partial \dot{q}_i}\,\dot{q}_i = 2\,T$$

As a result, equ. (27) yields

$$\Delta\int_{t_1}^{t_2} 2T\,dt = 0 \qquad (29)$$

Further if, there are no external forces acting on the system, as for example, a rigid body with no net applied forces, then T is conserved along with the total energy $H = E$. The least action principle from equ. (29) then takes the form

$$\Delta(t_2 - t_1) = 0 \qquad (30)$$

as T can be taken out from the integral.

The above equation states that of all the paths possible between two points, consistent with conservation of energy, the system moves along that particular path for which the time of transit is least. In this form the principle of least action recalls Fermat's principle in geometrical optics that a light ray travels between two points along such a path that the time taken is the least.

Still another form of the principle of least action was given by Carl Jacobi. To obtain it, let us write dt from the kinetic energy relation.

$$T = \tfrac{1}{2}\sum_i\sum_j m_{ij}\,\dot{q}_i\dot{q}_j = \tfrac{1}{2}\sum_i\sum_j m_{ij}\frac{dq_i}{dt}\frac{dq_j}{dt}$$

$$= \tfrac{1}{2}\,(d\rho/dt)^2,\ (d\rho)^2 = \sum_i\sum_j\ m_{ij}\,dq_i\,dq_j$$

or

$$dt = d\rho/\sqrt{2T} \qquad (31)$$

Inserting above into equ. (29), we get

$$\Delta\int_{\rho_1}^{\rho_2} \sqrt{2T}\,d\rho = 0$$

or finally

$$\Delta\int_{\rho_1}^{\rho_2} \sqrt{2[E - V(q)]}\, d\rho = 0 \tag{32}$$

This is referred to as the Jacobi's form of the principle of least action. ρ_1 and ρ_2 are the end points in the configuration space.

In terms of the cartesian coordinates (or in real space), the kinetic energy T of the system is given by

$$T = ½\sum_i m_i\left(\frac{ds_i}{dt}\right)^2$$

Substituting the value of dt from above into equ. (29) leads to

$$\Delta\int_{\rho_1}^{\rho_2} \sqrt{2[(E - V)\sum_i m_i ds_i^2} = 0 \tag{33}$$

where ρ_1 and ρ_2 are the positions of the system at times t_1 and t_2 in the real space.

For a single particle equ. (33) reduces to

$$\Delta\int_{\rho_1}^{\rho_2} \sqrt{2m[(E - V)}\; ds = 0 \tag{34}$$

Note that the above equation expresses the principle of least action in terms of the arc length of the particle trajectory.

PROBLEMS

6.1 Obtain equation of motion for a simple pendulum (see Fig. 6.2) by using Hamilton's equations of motion method.

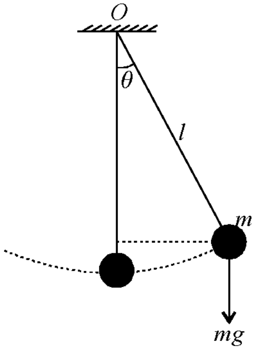


Fig. 6.2 Simple pendulum.

6.2 A particle of mass m is attracted toward a given point by a force of magnitude k/r^2 (central force), where k is a constant. Derive an expression for the Hamiltonian and Hamilton's equations of motion.

6.3 Write Hamiltonian for a charged particle in and electromagnetic field and there from obtain Hamilton's equations of motion.

6.4 Two springs having strengths k_1 and k_2 are joined and connected to a mass m on one side and to a rigid wall on the otherside as shown in Fig. 6.3. Using Hamilton's equations of motion show that the frequency of oscillation of m is

$\omega = [k_1 k_2/m\,(k_1 + k_2)]^{1/2}$

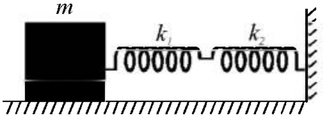


Fig. 6.3 Two springs having strengths k_1 and k_2 are joined and connected to a mass *m* on one side and to a rigid wall on the other side.

6.5 Two springs of strengths k_1 and k_2 are attached to a mass m and to fixed supports as shown in Fig. 6.4. Using Hamilton's equations of motion, show that the mass m oscillates with a frequency $\omega = [(k_1 + k_2)/m]^{1/2}$

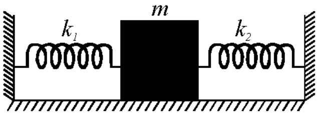


Fig. 6.4 Two springs of strengths k_1 and k_2 attached to a mass *m* and to fixed supports.

6.6 Consider a disk of mass *m* rolling without slipping down an inclined plane of length *l* and inclination ϕ as shown in Fig. 2.2. Derive an expression for the Hamiltonian and there from obtain the Hamilton's equations of motion.

SOLUTIONS

6.1 The kinetic and potential energies of the mass *m* (see Fig. 6.2) are given by

$$T = ½\, mv^2 = ½\, m\,(l\dot{\theta})^2$$

and

$$V = mg\,l\,(1 - \cos\theta)$$

Therefore, the Lagrangian

$$L = T - V = ½\, ml^2\,\dot{\theta}^2 - mgl(1 - \cos\theta)$$

and the conjugate momentum

$$p_\theta = \partial L/\partial\dot{\theta} = ml^2\dot{\theta} \qquad (35)$$

The Hamiltonian, in this case would be given by

$$H = \sum_i p_i\, \dot{q}_i - L = p_\theta \dot{\theta} - L$$
$$= \tfrac{1}{2}\, ml^2 \dot{\theta}^2 + mg\, l(1 - \cos\theta) \qquad (36)$$

which is noting but the total energy of the mass in this case.

Putting value of $\dot{\theta}$ from equ. (35) into equ. (36), we get

$$H = 1/2\; ml^2 p_\theta^2 + mg\, l(1 - \cos\theta)$$

As a result Hamilton's equations of motion are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = p_\theta / ml^2 \qquad (37)$$

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = \mathrm{mg}\; l \sin\theta \qquad (38)$$

Putting the value of $\dot{p}_\theta$ from equ. (37) into equ. (38), we find

$$\ddot{\theta} + g/l \sin\theta = 0$$

which for small θ becomes

$$\ddot{\theta} + g/l\, \theta = 0$$

This represents the equation of motion for a simple pendulum and is same as that obtained by using Lagrange's equations or by using Newtonian mechanics.

6.2 The kinetic and potential enegies of the mass m are given by

$$T = \tfrac{1}{2}\, m(\dot{r}^2 + r^2\, \dot{\theta}^2),$$

and $$V = -\int \vec{F} \cdot d\vec{r} = \int k/r^2\, dr = -k/r$$

Therefore, the Lagrangian

$$L = T - V = \tfrac{1}{2}\, m\, (\dot{r}^2 + r^2\, \dot{\theta}^2) + k/r \qquad (39)$$

As a result the congugate momenta are given by

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \text{ or } \dot{r} = p_r/m \qquad (40)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\, \dot{\theta} \text{ or } \dot{\theta} = p_\theta/mr^2 \qquad (41)$$

The Hamiltonian in this case would be given by

$$H = \sum_i p_i\, \dot{q}_i - L = p_r \dot{r} + p_\theta \dot{\theta} - L \qquad (42)$$

Putting the values of L, $\dot{r}$, and $\dot{\theta}$ from equs. (39) to (41) into equ. (42), Hamiltonian H becomes.

$$H = \tfrac{1}{2}m\left(p_r^2 + \frac{p_\theta^2}{r^2} \right) - k/r \qquad (43)$$

Therefore, the Hamilton's equations of motion are

$$-\dot{p}_r = \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{k}{r^2} \text{ or } \dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} \qquad (44)$$

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0 \text{ or } \dot{p}_\theta = 0 \text{ or } p_\theta = \text{Constant} \tag{45}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \text{ or } p_r = m\dot{r} \tag{46}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \text{ or } \mathrm{p}_\theta = \mathrm{mr}^2\,\dot{\theta} \tag{47}$$

One may note that equs. (46) and (47) are same as equs. (40) and (41) whereas equ. (45) gives the familiar constant of motion, $p_\theta = mr^2\,\dot{\theta}$ = constant.

6.3 Let us consider a particle of charge q and mass m, moving with a velocity $\vec{\upsilon}$ in an electromagnetic field. The Lagrangian of the particle would be given by

$$L = T - V = \tfrac{1}{2}\, m\upsilon^2 - q\phi + q\,(\vec{\upsilon}\,.\,\vec{A}\,)$$

$$= \tfrac{1}{2}\, m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\phi + q(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)$$

where $\vec{A}$ and ϕ denote the vector and scalar potentials, respectively.

As a result, the conjugate momenta are given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x,$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + qA_y,$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} + qA_z$$

From Lagrange's equation of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

Therefore, $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} = \dot{p}_x = 0$ as $\dfrac{\partial L}{\partial x} = 0$. This implies that p_x (and similarly the other components p_y and p_z of the mechanical momentum) is a constant. Effectively this means the term

$$\vec{p} = m\vec{\upsilon} + q\vec{A} \tag{48}$$

is a constant.

The Hamiltonian H is given by

$$H = \sum_i p_i\,\dot{q}_i - L$$

$$= \sum_i (mv_i + qA_i)\,v_i - [\tfrac{1}{2}\, mv^2 - q\phi - q\phi + q\,(\vec{\upsilon}\,.\vec{A})]$$

$$= \tfrac{1}{2}\, mv^2 + q\phi$$

$$= \frac{1}{2m}(\vec{p} - q\vec{A}) \cdot (\vec{p} - q\vec{A}) + q\phi \tag{49}$$

where use of equ. (48) has been made

The Hamilton's equations of motion are, therefore,

$$\dot{\vec{r}} = \partial H/\partial \vec{p} = 1/m(\vec{p} - q\vec{A}) = \vec{\upsilon}$$

where use has been made of equs. (49) and (48). And

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = -\nabla H = -q\nabla\phi + q\nabla(\vec{\upsilon} \cdot \vec{A})$$

as $\vec{p}$ does not depend on space coordinates $\vec{r}$.

Note that the product $(\vec{p} - q\vec{A}) . (\vec{p} - q\vec{A})$ has been differentiated and finally $(\vec{p} - q\vec{A}) = m\vec{\upsilon}$ has been put in obtaining the above.

6.4 Let x_1 and x_2 represent the displacements from the equilibrium positions of the springs with strengths k_1 and k_2 respectively. In that case the kinetic and potential energies of the system (Fig. 6.3) would be given by

$$T = ½\, m\dot{x}_1^2 + ½\, m\dot{x}_2^2$$

and

$$V = ½\, k_1 x_1^2 + ½\, k_2 x_2^2$$

where $x_1 + x_2 = x$ is the combined displacement (extension).

Therefore, the Lagrangian

$$L = T - V = ½m\dot{x}_1^2 + ½m\dot{x}_2^2 - ½k_1x_1^2 - ½k_2x_2^2$$

As a result, the conjugate momenta are

$$p_{x_1} = \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1, p_{x_2} = \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$$

The Hamiltonian in this case would be given by

$$H = p_{x_1}\dot{x}_1 + p_{x_2}\dot{x}_2 - L$$

$$= p_{x_1}^2/2m + p_{x_2}^2/2m + ½k_1x_1^2 + ½\,k_2x_2^2$$

Therefore, the Hamilton's equations of motion are

$$\dot{x}_1 = \frac{\partial H}{\partial p_{x_1}} = p_{x1}/m, \dot{x}_2 = \frac{\partial H}{\partial p_{x_2}} = p_{x2}/m,$$

$$-\dot{p}_{x1} = \frac{\partial H}{\partial x_1} = k_1\, x_1, -\dot{p}_{x2} = \frac{\partial H}{\partial x_2} = k_2\, x_2$$

If F is the applied force, then

$$F = \dot{p}_{x1} = \dot{p}_{x2}$$

and hence $F/k_1 + F/k_2 = -(x_1 + x_2) = -x$

or $$F = \frac{-k_1 k_2}{k_1 + k_2} x$$

Therefore, the acceleration

$$a = F/m = \frac{k_1 k_2}{(k_1 \quad k_2)} x = -\omega^2 x$$

which amounts to the frequency of oscillation of m,

$$\omega = [k_1 k_2 / m(k_1 + k_2)]^{1/2}$$

6.5 Let x be the extension in the spring with strength k_1 and hence x would be the compression in the spring with strength k_2. In that case the kinetic and potential energies of the system would be given by

$$T = ½\, m\dot{x}^2$$

and $$V = ½\, k_1 x^2 + ½\, k_2 x^2$$

Therefore, the Lagrangian

$$L = T - V = ½\, m\dot{x}^2 - ½\,(k_1 + k_2)x^2$$

and hence the conjugate momentum

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{50}$$

The Hamiltonian in this case would be

$$\begin{aligned} H &= p_x \dot{x} - L = m\,\dot{x}\,\dot{x} - ½\, m\dot{x}^2 + ½(k_1 + k_2)\, x^2 \\ &= ½\, m\dot{x}^2 + ½(k_1 + k_2)\, x^2 \end{aligned} \tag{51}$$

Putting the value of $\dot{x}$ from equ. (50) into equ. (51), we get

$$H = p_x^2/2\, m + ½\,(k_1 + k_2)\, x^2$$

As a result the Hamilton's equations of motion are

$$\dot{x} = \frac{\partial L}{\partial p_x} = p_x / m, \tag{52}$$

$$-\dot{p}_x = \frac{\partial H}{\partial x} = (k_1 + k_2)\, x \tag{53}$$

Putting the value of $\dot{p}_x$ from equ. (52) into (53), we get

$$\ddot{x} + (k_1 + k_2)/m\, x = 0$$

which implies that the mass oscillates with frequency

$$\omega = [(k_1 + k_2)/m]^{1/2}$$

6.6 The kinetic energy of the disk consists of translational and rotational energies and is given by

$$\begin{aligned} \mathrm{T} &= ½\, m\dot{x}^2 + ½\, I\dot{\theta}^2 \\ &= ½\, m\dot{x}^2 + ¼\, mR^2\, \dot{\theta}^2 \end{aligned}$$

where $I = ½\, mR^2$ is the moment of inertin of the disk about its axis of rotation. The potential energy of the disk

$$V = mg(l - x)\sin\phi$$

where l is the length of the inclined plane and zero of the potential energy is taken at $x = l$. The Lagrangian of the system is, therefore

$$L = T - V = ½m\dot{x}^2 + ¼mR^2\dot{\theta}^2 - mg(l - x)\sin\phi \tag{54}$$

Since the disk rolls without slipping, the constraint on the problem is

$$x = R\theta \tag{55}$$

which amounts to the conclusion that the system has only one degree of freedom, we can therefore choose either x or θ as our generalized coordinate and use equ. (29) to eliminate the other.

Substituting equ. (55) into equ. (54), we get

$$\mathrm{L} = ¾\, m\dot{x}^2 - mg(l - x)\sin\phi \tag{56}$$

As a result the conjugate or generalized momentum,

$$p_x = \frac{\partial L}{\partial \dot{x}} = 3/2\ \mathrm{m}\dot{x} \tag{57}$$

Using equs. (56) and (57), the Hamiltonian of the system

$$\mathrm{H} = p_x\,\dot{x} - L = p_x^2/3m + mg(l - x)\sin\phi$$

Therefore, the Hamilton's equations of motion are

$$\dot{x} = \frac{\partial L}{\partial p_x} = 2p_x/3m \tag{58}$$

and
$$-\dot{p}_x = \frac{\partial H}{\partial x} = -\,mg\sin\phi \tag{59}$$

Taking time derivative of equ. (58) and then eliminating $\dot{p}_x$ from the resulting equation and equ. (59), we find

$$\ddot{x} = 2/3\ g\sin\phi \tag{60}$$

Differentiating equ. (55) twice, we get

$$\ddot{x} = R\ddot{\theta} \tag{61}$$

Then substituting $\ddot{x}$ from equ. (60) into above, we get

$$\ddot{\theta} = 2g\sin\phi/3R \tag{61}$$

7

CHAPTER

Canonical Transformations

7.1 CANONICAL (OR CONTACT) TRANSFORMATIONS

Suppose a given system can be described by more than one set of generalized coordinates. For example, to discuss the motion of a particle in a plane, one may use as generalized coordinates either the cartesian coordinates

$$q_1 = x, \quad q_2 = y$$

or the plane polar coordinates

$$q_1 = r, \quad q_2 = \theta$$

It may be noted that both the choices are equally valid, but one or the other set may be more convenient for the problem under discussion. Notice that for central forces neither x nor y is cyclic whereas the second set does contain a cyclic coordinate θ. Therefore, by choosing second set, solution of the problem becomes easier as only one variable r is to appear in the Hamiltonian. Consequently we want to discuss here a specific procedure for transforming from one set of variables to some other set which may be more suitable or in which some of the coordinates are cyclic.

The transformation equations from cartesian to plane polar coordinates read as

$$\begin{aligned} r &= (x^2 + y^2)^{1/2} = r\,(x, y) \\ \theta &= \tan^{-1} y/x = \theta\,(x, y) \end{aligned} \qquad (1)$$

One can generalize the above transformation in the configuration space and can go from one set of coordinates, (or variables) q_i to a new set of coordinates Q_i by transformation equations of the form

$$Q_i = Q_i(q_1, q_2, \ldots\ldots\ldots\ldots q_n, t)\ (i = 1, 2, \ldots\ldots\ldots\ldots, n) \qquad (2)$$

where it is assumed that variable t appears explicitly. Such transformations are known as point transformations in the configuration space.

Passing now to the Hamiltonian formalism, we must recall that the generalized coordinates and the generalized momenta are independent variables. This means that in the phase space the most general (invertible) transformation is

$$Q_i = Q_i(q, p, t), P_i = P_i(q, p, t)\ (i = 1, 2, \ldots\ldots\ldots, n) \qquad (3)$$

where we have left out the indices inside the parentheses. This clearly is a point transformation in the 2n - dimensional phase space.

If the system of canonical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \ \dot{p}_i = -\frac{\partial H}{\partial q_i} \ (i = 1, 2, ..., n)$$

keep their form under the transformation (3), that is, if the new variables Q_i, P_i obey

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \ \dot{P}_i = -\frac{\partial K}{\partial Q_i} \ (i = 1, 2, ..., n) \tag{4}$$

where

$$K = K(Q_1 ....Q_n, P_1, ...P_n, t) \tag{5}$$

is the Hamiltonian of the system in the new representation, the transformation (3) preserving the form of Hamilton's equations of motion is called canonical.

Let us now find the condition of canonicity. To this end, we use modified Hamilton's principle

$$\delta \int_{t_1}^{t_2} (\sum_i p_i \dot{q}_i - H(q, p\, t))\, dt = 0 \tag{6}$$

We are already acquainted with the fact that the canonical equations can be derived by means of modified Hamilton's principle. In order that the new variables Q_i and P_i are to be canonical (or they satisfy canonical equations), they must verify (or satisfy) modified Hamilton's principle

$$\delta \int_{t_1}^{t_2} (\sum_i P_i \dot{Q}_i - K(Q, P, t))dt = 0 \tag{7}$$

The simultaneous validity of equs. (6) and (7) does not mean that the integrals in both expressions are equal. They can differ only by a total time derivative of some function F, because

$$\delta \int_{t_1}^{t_2} \frac{dF}{dt} dt = \delta F(t_2) - \delta F(t_1) = 0$$

as in the case of the modified Hamilton's principle, there are no variations at the end points.

Consequently, we obtain

$$(\sum_i p_i \dot{q}_i - H(q, p, t)) - (\sum_i P_i \dot{Q}_i - K(Q, P, t)) = \frac{dF}{dt} \tag{8}$$

known as the canonicity condition or in other words if the above condition is satisfied, the transformation is canonical.

The function F is called the generating function of the canonical transformation (3). Note that the first bracketed term of equ. (8) is function of q,p, and t and the second is a function of Q, P, and t. F is, therefore, in general a function of $(4n+1)$ variables q_i, p_i, Q_i, P_i and t. However, there are $2n$ transformation equations (3), therefore out of $4n$ variables besides t, only $2n$ are independent. Thus F is a function of $2n+1$ independent variables.

Note that the generating function F makes possible the transition from the old variables q_i, p_i to the new variables Q_i, P_i. It must depend on n old variables (either q_i or p_i), n new variables (either Q_i or P_i), and, of course, on time t. Thus there are four possibilities for F,

$$F_1(q, Q, t),\ F_2(q, P, t),\ F_3(p, Q, t), F_4(p, P, t)$$

If $F = F_1(q, Q, t)$, equ. (8) then takes the form

$$(\sum_i p_i\, \dot{q}_i - H) - (\sum_i P_i\, \dot{Q}_i - K) = \frac{d}{dt} F_1(q, Q, t) \tag{9}$$

or
$$(\sum_i p_i\, \dot{q}_i - H) - (\sum_i P_i\, \dot{Q}_i - K) = \sum_i \left(\frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i \right) + \frac{\partial F_1}{\partial t}$$

or
$$\sum_i \left(p_i - \frac{\partial F_1}{\partial q_i} \right) \dot{q}_i - \sum_i \left(P_i - \frac{\partial F_1}{\partial Q_i} \right) \dot{Q}_i + K - H - \frac{\partial F_1}{\partial t} = 0 \tag{10}$$

Since the old and the new coordinates q_i and Q_i are separately independent, equ. (10) can hold identically only if the coefficients of $\dot{q}_i$ and $\dot{Q}_i$ each vanish. That is, if

$$p_i = \frac{\partial F_1}{\partial q_i}, \tag{11a}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i}, \tag{11b}$$

$$K = H + \frac{\partial F_1}{\partial t} \tag{11c}$$

with $i = 1, 2, \ldots\ldots\ldots, n$

Equs. (11a) give p_i as functions of q_i, Q_i, and t. Assuming they can be inverted, they could then be solved for the Q_i's in terms of q_i, p_i, and t, thus yielding the first set $(3)_1$ of the canonical transformations. These are then substituted into equs. (11b) so that they give P_i's as functions of q_i, p_i and t, that is the second set $(3)_2$ of the canonical transformations. Finally, equ. (11c) provides the connection between the two Hamiltonians, H and K.

It sometimes happens that it is not suitable to describe the canonical transformation by generating function of the type $F_1(q, Q, t)$ instead it is function of q, P, and t, that is $F = F_2\,(q, P, t)$. The transition from q_i, Q_i to q_i, P_i is accomplished by means of a Legendre transformation (See Appendix B).

Putting $u = -P_i$, $x = Q_i$, $y = q_i$, $g = F_2$, $f = F_1$ in equ. (B.5), we get

$$F_2(q, P, t) = F_1(q, Q, t) + \sum_i P_i Q_i \tag{12}$$

Substituting F_1 from above into equ. (9), we get

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{d}{dt}[F_2(q, P, t) - \sum_i P_i Q_i]$$

$$= -\sum_i Q_i \dot{P}_i - K + \frac{d}{dt} F_2(q, P, t) \tag{13}$$

Again following the same procedure given in the previous case, we obtain

$$p_i = \frac{\partial F_2}{\partial q_i}, \tag{14a}$$

$$Q_i = \frac{\partial F_2}{\partial q_i}, \tag{14b}$$

$$K = H + \frac{\partial F_2}{\partial t} \tag{14c}$$

with $i = 1, 2, \ldots\ldots\ldots, n$

As before, equs. (14*a*) can be solved for P_i as functions of q_i, p_i, and t to correspond to the second half of the transformation equs. (3). The remaining half of the transformation equations is then provided by equs. (14*b*).

To connect $F_3(p, Q, t)$ with $F_1(q, Q, t)$, we again apply Legendre transformation. For this case, we put $u = p_i$, $x = q_i$, $y = Q_i$, $g = F_3$, $f = F_1$ in equ. (B.5) resulting in

$$F_3(p, Q, t) = F_1(q, Q, t) - \sum_i p_i q_i \tag{15}$$

Substituting F_1 from above into equ. (9) and repeating the procedure used for obtaining equs. (11), we get

$$q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}, \tag{16a}$$

$$K = H + \frac{\partial F_3}{\partial t} \quad (i = 1, 2, \ldots\ldots, n) \tag{16b}$$

with the transformation equs. (3) being derived from the equs. (16*a*).

To connect $F_4(p, P, t)$ with $F_1(q, Q, t)$, we perform double Legendre transformation (one first goes from F_1 to F_2 and then from F_2 to F_4) resulting in

$$F_4(p, P, t) = F_1(q, Q, t) + \sum_i P_i Q_i - \sum_i p_i q_i \tag{17}$$

Substituting F_1 from above into equ. (9) and repeating the procedure used for obtaining equs. (11), we get

$$q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i}, \tag{18a}$$

$$K = H + \frac{\partial F_4}{\partial t} \quad (i = 1, 2, \ldots\ldots, n) \tag{18b}$$

with the transformation equs. (3) being derived from the equs. (18*a*).

7.2 EXAMPLES AND APPLICATIONS OF CANONICAL TRANSFORMATIONS

The following simple yet important examples would provide the reader a better idea of the utility of canonical transformations.

1. Consider the generating function

$$F = F_2 = \sum_i q_i P_i \tag{19}$$

From equs. (14), the transformation equations are

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i,\ Q_i = \frac{\partial F_2}{\partial P_i} = q_i,\ K = H\ (i = 1, 2, \ldots.., n) \tag{20}$$

implying that the new and old coordinates are the same. Therefore, F_2 generates the identify transformation.

Now, if instead we choose

$$F = F_2 = -\sum_i q_i P_i \tag{21}$$

then

$$p_i = -P_i,\ Q_i = -q_i,\ K = H \tag{22}$$

Therefore, any inversion in the phase space is a canonical transformation.

2. Consider the generating function

$$F = F_1 = \sum_i q_i Q_i \tag{23}$$

The corresponding transformation equations from equs. (11) are

$$p_i = \frac{\partial F_1}{\partial q_i} = Q_i,\ P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i,\ K = H \tag{24}$$

Implying that the transformation interchanges the momenta and coordinates. The new coordinates are the old momenta and the new momenta are the old coordinates. Therefore, F_1 generates the exchange transformation.

3. Consider a canonical transformation that can be used to solve the problem of a simple one-dimensional harmonic oscillator. The Hamiltonian for the given problem in terms of the usual coordinates is

$$H = p^2/2m + \tfrac{1}{2} kq^2$$
$$= p^2/2m + \tfrac{1}{2} m\omega^2 q^2$$

where $\omega^2 = k/m$, k, m, and ω being the force constant, mass and angular frequency of the harmonic oscillator, respectively.

Consider now the following generating function

$$F = F_1 = \tfrac{1}{2} m\omega q^2 \cot Q \tag{25}$$

Using equs. (11), we find

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q, \tag{26}$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega}{2\sin^2 Q} q^2, \tag{27}$$

$$K = H \tag{28}$$

The desired transformation is, therefore

$$q = (2P/m\omega)^{1/2} \sin Q, \tag{29a}$$

and on putting above into equ. (26),

$$p = (2\, m\omega P)^{1/2} \cos Q \tag{29b}$$

and hence

$$K = H = p^2/2m + \tfrac{1}{2} m\omega^2 q^2 = \omega P \tag{29c}$$

Thus the Hamiltonian is cyclic in Q, which means the conjugate momentum

$$P = \text{Const.}$$

From equs. (4) and (29*c*), the equation of motion for Q reduces to the simple form

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega$$

which amounts to

$$Q = \omega t + \alpha$$

where α is a constant of integration. Using equ. (29*a*) and above, we get

$$q = (2\, E/m\omega)^{1/2} \sin(\omega t + \alpha)$$

where we have put $P = E/\omega$ from equ. (29c), that is the constant energy divided by ω, giving the equation for simple harmonic motion.

This provides us a simple example of how the Hamiltonian can be reduced to a form cyclic in all coordinates by means of canonical transformations.

7.3 POISSON BRACKETS

Let $f(q, p, t)$ and $g(q, p, t)$ represent any two functions of canonical variables q_1, q_n, p_1, ... , p_n and of time t. The expression

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{32}$$

is called the Poisson bracket of f and g.

By means of the above notation and from Hamilton's (or canonical) equations,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \; \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

we can write the total time derivative of any function $f(q, p, t)$ as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right)$$

$$= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

or

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\} \tag{33}$$

If f does not depend on time t explicitly, the above equation reduces to

$$\frac{df}{dt} = \{f, H\} \tag{34}$$

Further if f is a constant (or integral) of motion ($df/dt = 0$), the above equation reduces to

$$\{f, H\} = 0, \tag{35}$$

that is, the Poisson bracket of the integral f with the Hamiltonian H vanishes. This implies that all functions whose Poisson brackets with the Hamiltonian vanish, would therefore be constants of motion. We thus have a general test for seeking and identifying the constants of motion of the system.

The following properties result immediately from the definition (32) of the Poisson bracket:

(i) $\{f, g\} = -\{g, f\}$

(ii) $\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$

(iii) $\{f_1 f_2, g\} = f_1 \{f_2, g\} + (f_1, g) f_2$

(iv) $\{c, f\} = 0$, c = constant

(v) $\{f, cg\} = c\{f, g\}$

7.4 SOME MORE PROPERTIES OF POISSON BRACKETS

(i) $\{q_i, q_k\} = 0$, $\{p_i, p_k) = 0$, $\{q_i, p_k\} = \delta_{ik}$

Putting $f = q_i$ in equ. (32), we get

$$\{q_i, g\} = \frac{\partial g}{\partial p_i} \tag{36}$$

Similarly, if we put $f = p_i$ in equ. (32), we get

$$\{p_i, g\} = -\frac{\partial g}{\partial q_i} \tag{37}$$

Further putting $g = q_k$ in (36) and $g = p_k$ in (37), we get

$$\{q_i, q_k\} = 0, \{p_i, p_k\} = 0 \tag{38}$$

From equ. (32),

$$\{q_i, p_k\} = \sum_i \left(\frac{\partial q_i}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \frac{\partial q_i}{\partial p_i}\frac{\partial p_k}{\partial q_i} \right) = \frac{\partial p_k}{\partial p_i} = \delta_{ik} \tag{39}$$

where δ_{ik} is the Kronecker delta function. Thus

$$\{q_i, q_k\} = 0, \{p_i, p_k\} = 0, \{q_i, p_k\} = \delta_{ik} \tag{40}$$

(ii) Poisson Brackets are Invariant Under Canonical Transformation

Let f and g be any two arbitrary functions, therefore

$$\{f, g\}_{q,p} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \tag{41}$$

Assuming that q_i, p_i are functions of new variables Q_k, P_k, equ. (41) as a result becomes

$$\{f, g\}_{q,p} = \sum_{ik} \left[\frac{\partial f}{\partial q_i}\left(\frac{\partial g}{\partial Q_k}\frac{\partial Q_k}{\partial p_i} + \frac{\partial g}{\partial P_k}\frac{\partial P_k}{\partial p_i} \right) - \frac{\partial f}{\partial p_i}\left(\frac{\partial g}{\partial Q_k}\frac{\partial Q_k}{\partial q_i} + \frac{\partial g}{\partial P_k}\frac{\partial P_k}{\partial q_i} \right) \right]$$

$$= \sum_k \left[\frac{\partial g}{\partial Q_k}\{f, Q_k\}_{q,p} + \frac{\partial g}{\partial P_k}\{f, P_k\}q,p \right] \tag{42}$$

Using equs. (36) and (37), that is

$$\{Q_k, f\} = \frac{\partial f}{\partial P_k} \tag{43}$$

and

$$\{P_k, f\} = -\frac{\partial f}{\partial Q_k} \tag{44}$$

equ. (42) takes the form

$$\{f, g\}_{q,p} = \sum_k \left(-\frac{\partial g}{\partial Q_k}\frac{\partial f}{\partial P_k} + \frac{\partial g}{\partial P_k}\frac{\partial f}{\partial Q_k} \right) = \{f, g\}_{Q,P} \tag{45}$$

which shows that the Poisson bracket is invariant under canonical transformation in the phase space. Hence, there is no need of writing the subscripts (q, p) or (Q, P) on the Poisson bracket.

(iii) Hamilton's Equations of Motion in Poisson Bracket Notation

From equ. (34), the time derivative of any function f which is not an explicit function of time t,

$$\frac{df}{dt} = \{f, H\} \tag{34}$$

Putting $f = q_i$ in the above equation, we get

$$\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i} \tag{46}$$

where we have made use of equ. (36).

Further putting $f = p_i$ in equ. (34), we get

$$\dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i} \tag{47}$$

where we have made use of equ. (37).

Equs. (46) and (47) are Hamilton's equations of motion in Poisson Bracket notation.

(iv) Jacobi's Identity

To verify the validity of Jacobi's identity

$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0,$

let us calculate:

$\{f, \{g, h\}\} - \{g, \{f, h\}\} =$

$$\left\{f, \sum_i \left(\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\right)\right\} - \left\{g, \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i}\right)\right\}$$

In view of the property

$\{f, g_1 g_2\} = g_1 \{f, g_2\} + g_2 \{f, g_1\},$

$$\begin{aligned}\{f, \{g, h\}\} - \{g, \{f, h\}\} = \sum_i \Bigg[& \frac{\partial g}{\partial q_i}\left\{f, \frac{\partial h}{\partial p_i}\right\} + \frac{\partial h}{\partial p_i}\left\{f, \frac{\partial g}{\partial q_i}\right\} \\ & - \frac{\partial g}{\partial p_i}\left\{f, \frac{\partial h}{\partial q_i}\right\} - \frac{\partial h}{\partial q_i}\left\{f, \frac{\partial g}{\partial p_i}\right\} \\ & - \frac{\partial f}{\partial q_i}\left\{g, \frac{\partial h}{\partial p_i}\right\} - \frac{\partial h}{\partial p_i}\left\{g, \frac{\partial f}{\partial q_i}\right\} \\ & + \frac{\partial f}{\partial p_i}\left\{g, \frac{\partial h}{\partial q_i}\right\} + \frac{\partial h}{\partial q_i}\left\{g, \frac{\partial f}{\partial p_i}\right\} \Bigg]\end{aligned}$$

$$= \sum_i \frac{\partial h}{\partial p_i}\Big[\Big\{f, \frac{\partial g}{\partial q_i}\Big\} + \Big\{\frac{\partial f}{\partial q_i}, g\Big\}$$

$$- \sum_i \frac{\partial h}{\partial q_i}\Big[\Big\{f, \frac{\partial g}{\partial p_i}\Big\} + \Big\{\frac{\partial f}{\partial p_i}, g\Big\}\Big]$$

$$+ \sum_i \Big[\frac{\partial g}{\partial q_i}\Big\{f, \frac{\partial h}{\partial p_i}\Big\} - \frac{\partial g}{\partial p_i}\Big\{f, \frac{\partial h}{\partial q_i}\Big\}$$

$$- \frac{\partial f}{\partial q_i}\Big\{g, \frac{\partial h}{\partial p_i}\Big\} + \frac{\partial f}{\partial p_i}\Big\{g, \frac{\partial h}{\partial q_i}\Big\}\Big]\Big]$$

We would now show that the third sum in the above expression vanishes. That is,

$$\sum_i \Big[\frac{\partial g}{\partial q_i}\Big\{f, \frac{\partial h}{\partial p_i}\Big\} - \frac{\partial g}{\partial p_i}\Big\{f, \frac{\partial h}{\partial q_i}\Big\} - \frac{\partial f}{\partial q_i}\Big\{g, \frac{\partial h}{\partial p_i}\Big\} + \frac{\partial f}{\partial p_i}\Big\{g, \frac{\partial h}{\partial q_i}\Big\}\Big] = 0$$

Again using the definition of Poisson bracket the r.h.s. of the above expression becomes

$$\sum_{ij}\Big[\frac{\partial g}{\partial q_i}\Big\{\frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j}\Big(\frac{\partial h}{\partial p_i}\Big) - \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q_j}\Big(\frac{\partial h}{\partial p_i}\Big)\Big\}\Big]$$

$$- \sum_{ij}\Big[\frac{\partial g}{\partial p_i}\Big\{\frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j}\Big(\frac{\partial h}{\partial q_i}\Big) - \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q_j}\Big(\frac{\partial h}{\partial q_i}\Big)\Big\}\Big]$$

$$- \sum_{ij}\Big[\frac{\partial f}{\partial q_i}\Big\{\frac{\partial g}{\partial q_j}\frac{\partial}{\partial p_j}\Big(\frac{\partial h}{\partial p_i}\Big) - \frac{\partial g}{\partial p_j}\frac{\partial}{\partial q_j}\Big(\frac{\partial h}{\partial p_i}\Big)\Big\}\Big]$$

$$+ \sum_{ij}\Big[\frac{\partial f}{\partial p_i}\Big\{\frac{\partial g}{\partial q_j}\frac{\partial}{\partial p_j}\Big(\frac{\partial h}{\partial q_i}\Big) - \frac{\partial g}{\partial p_j}\frac{\partial}{\partial q_j}\Big(\frac{\partial h}{\partial q_i}\Big)\Big\}\Big]$$

Now taking into account the fact that the order of differentiation is unimportant and the fact that the sum is not affected if the indices are interchanged (changing i to j and j to i) as they are dummy indices, the r.h.s. becomes zero.

Further using the identity

$$\partial/\partial x\{f, g\} = \{\partial f/\partial x, g\} + \{f, \partial g/\partial x\},$$

one finds,

$$\{f, \{g, h\}\} - \{g, \{f, h\}\} = \sum_i \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i}\{f, g\} - \frac{\partial h}{\partial q_i}\frac{\partial}{\partial p_i}\{f, g\} + 0$$

$$= -\{h, \{f, g\}\}$$

where once again we have used the definition of Poisson bracket $\{h, \{f, g\}\}$. Therefore, we get

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \qquad (48)$$

which completes the proof.

(v) Poisson Theorem. If f and g are two constants (integrals) of motion, their Poisson bracket {f, g} is also an integral of motion, that is

$$\{f, g\} = \text{const}$$

which is Poisson theorem.

The proof of the theorem is very simple if f and g do not depend explicitly on time. Putting h = H in Jacobi's identity, we obtain

$$\{f, \{g, H\}\} + \{H, \{f, g\}\} + \{g, \{H, f\}\} = 0 \qquad (49)$$

The first and third term of the identity vanish because the Poisson bracket of any constant of motion with H vanishes. We are, therefore, left with

$$\{H, \{f, g\}\} = 0$$

which means $\{f, g\}$ is a constant of motion. This is the required result

If the integrals (or constant) of motion f and g are explicitly time dependent, then from equ. (33),

$$\frac{\partial}{\partial} + \{f, H\} = 0, \ \frac{\partial g}{\partial t} + \{g, H\} = 0 \qquad (50)$$

Using above, equ. (49) takes the form

$$\left\{f, -\frac{\partial g}{\partial t}\right\} + \{H, \{f, g\}\} + \left\{g, \frac{\partial f}{\partial t}\right\} = 0$$

or $$\{\frac{\partial}{\partial}, g\} + \{f, \frac{\partial g}{\partial t}\} + \{\{f, g\}, H\} = 0$$

In view of the identity

$$\frac{\partial}{\partial t}\{f, g\} = \{\frac{\partial f}{\partial t}, g\} + \{f, \frac{\partial g}{\partial t}\},$$

the above equation reduces to

$$\frac{\partial}{\partial t}\{f, g\} + \{\{f, g\}, H\} = 0$$

which amounts to the conclusion that $\partial/\partial t\{f, g\} = 0$ or $\{f, g\}$ is constant of motion. Thus Poisson's theorem is proved

7.5 INFINITESIMAL CANONICAL (OR CONTACT) TRANSFORMATIONS

Any canonical transformation that differs infinitesimally from the identity transformation (see equ. 20), is called infinitesimal canonical transformation. One may recollect that

$$F_2(q, P) = \sum_i q_i P_i$$

is the generating function of the identity transformation

$$Q_i = q_i, P_i = p_i \qquad (i = 1, 2, \ldots\ldots., n)$$

If the canonical variables Q_i, P_i differ from the old coordinates, (or variables) q_i, p_i by infinitesimal variations δq_i, δp_i, then

$$Q_i = q_i + \delta q_i, P_i = p_i + \delta p_i \qquad (i = 1, 2, \ldots\ldots, n) \qquad (51)$$

is an infinitesimal canonical transformation. It is obvious that the generating function in the present case would also differ only by an infinitesimal amount from the one corresponding to the identity transformation, that is, F_2. We can, therefore, write the generating function in the case under discussion as

$$F_2' = \sum_i q_i P_i + \in G(q_i, P_i) \qquad (52)$$

where G is an arbitrary function of q_i, P_i and $\in$ is an infinitesimal parameter, independent of canonical variables. Using transformation equs. (20), we get

$$p_i = \frac{\partial F_2'}{\partial q_i} = P_i + \in \frac{\partial G}{\partial q_i}, Q_i = \frac{\partial F_2'}{\partial P_i} = q_i + \in \frac{\partial G}{\partial P_i} \qquad (53)$$

Thus the infinitesimal variations in equs. (51) become

$$Q_i - q_i = \delta q_i = \in \frac{\partial G}{\partial P_i}, P_i - p_i = \delta p_i = -\in \frac{\partial G}{\partial q_i} \quad (i = 1, 2, \ldots, n) \quad (54)$$

Since P_i is only slightly different from p_i, we can take $G(q_i, p_i)$ instead of $G(q_i, P_i)$ and also $\frac{\partial G}{\partial p_i}$ instead of $\frac{\partial G}{\partial P_i}$. Therefore,

$$\delta q_i = \in \frac{\partial G}{\partial p_i}, \delta p_i = -\in \frac{\partial G}{\partial q_i} \quad (i = 1, 2, \ldots\ldots., n) \qquad (55)$$

in which $G(q_i, p_i)$ plays the role of the generating function of the infinitesimal canonical transformation, whereas δq_i, δp_i stand for the canonical variables.

As an example, let us consider a particular case in which $G = H(q_i, p_i)$, that is, the Hamiltonian taken as the generating function and $\in = dt$. Then from equ. (55),

$$\delta q_i = dt \frac{\partial H}{\partial p_i} = dt\, \dot{q}_i = dq_i,$$

$$\delta p_i = -dt \frac{\partial H}{\partial q_i} = dt\, \dot{p}_i = dp_i \qquad (56)$$

where use of the Hamilton's canonical equations of motion has been made.

Therefore, H generates an infinitesimal canonical transformation which changes the variables q_i, p_i at time t to the values they would have at time $t + dt$. Hence, the motion of the system in a time interval dt can be described by an infinitesimal transformation generated by the Hamiltonian. Accordingly, the motion of a system in a finite interval of time would be described as a succession of infinitesimal transformations. We, therefore, conclude that Hamiltonian is the generator of the motion of the system with time or generator of the time evolution.

7.6 RELATION BETWEEN INFINITESIMAL CONTACT TRANSFORMATIONS AND POISSON BRACKETS

Let us consider a function $u(q_i, p_i)$. The change in u as result of an infinitesimal canonical transformation is

$$\begin{aligned}\delta u &= u(Q_i, P_i) - u(q_i, p_i)\\ &= u(q_i + \delta q_i, p_i + \delta p_i) - u\,(q_i, p_i)\end{aligned}$$

Upto the first order of the infinitesimal parameters, using the Taylor's series expansion, one can write

$$\delta u = \sum_i \left(\frac{\partial u}{\partial q_i} \delta q_i + \frac{\partial u}{\partial p_i} \delta p_i \right)$$

In case the said transformation is generated by the function G, we can make use of equs. (55) and write

$$\begin{aligned}\delta u &= \in \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial G}{\partial q_i} \right)\\ &= \in \{u, G\} \qquad (57)\end{aligned}$$

Now if we put u = H, we get the change in the Hamiltonian as a result of the infinitesimal transformation as

$$\delta H = \in \{H, G\} \qquad (58)$$

It has already been shown that if $G(q, p)$ is a constant of motion, then its Poisson bracket with H, that is, $\{H, G\}$ must vanish or that means $\delta H = 0$ or $H =$ constant. Thus, a constant of motion generates an infinitesimal canonical transformation which leaves H unaffected. In other words, the constants of motion are generating functions of those infinitesimal canonical transformations which leave the Hamiltonian H unaffected or invariant. This conclusion implies a connection between the symmetry properties and the constants of motion or the conserved quantities.

The symmetry properties of the system indicate that which are the transformations that do not change the value of the Hamiltonian. Clearly, if the physical system is symmetrical under a given operation, the Hamiltonian must remain unaffected under the corresponding transformation. One can, therefore, determine all the constants of motion by examining the symmetry properties of the Hamiltonian.

As an illustration, consider the momentum conservation theorem. If a coordinate q_i is cyclic, the Hamiltonian then would be independent of q_i and would be invariant under an infinitesimal transformation that involves a displacement of q_i alone. Consider, now, a transformation generated by the generalized momentum (generating function in this case) conjugate to q_i:

$$G(q_i, p_i) = p_i \tag{59}$$

From equs. (55), the resultant canonical transformation is

$$\delta q_j = \in \delta_{ij} \text{ (for } i = j, \delta q_i = \in \text{ otherwise } \delta q = 0), \tag{60}$$

$$\delta p_j = 0$$

that is, exactly the required infinitesimal displacement $\in$ of q_i and only q_i. This proves the familiar momentum conservation theorem, that is, if a coordinate is cyclic, its conjugage momentum is a constant of motion or is conserved.

PROBLEMS

7.1 Show directly that the transformation

$Q = \log (1/q \sin p), p = q \cot p$

is canonical

7.2 Show directly that the transformation

$P = ½ (p^2 + q^2), Q = \tan^{-1} (q/p)$

is canonical

7.3 Show directly that the transformation

$Q = 1/p, P = qp^2$

is canonical

7.4 Show that $F_3 = \sum_i p_i Q_i$ generates reflection trasnformation.

7.5 The transformation equations between two sets of coordinates are

$Q = \log (1 + q^{1/2} \cos p)$,

$P = 2(1 + q^{1/2} \cos p)q^{1/2} \sin p$

a) Show directly from these transformation equations that Q, P are canonical variables if q and p are.

b) Show that the function that generates this transformation is

$F_3 = -(e^Q - 1)^2 \tan p$

7.6 The transformation equations between two sets of coordinates are

$P = q^\alpha \sin \beta p$,

$Q = q^\alpha \cos \beta p$

If the transformations are canonical find α and β. Also show that the function that generates this transformation is

$F_3 = -½ Q^2 \tan 2p$

7.7 Show directly that the transformation

$$Q = (2q)^{1/2} \cos p, \; P = (2q)^{1/2} \sin p$$

is canonical. Also show that the function which generates this transformation is

$$F_1 = q \cos^{-1} Q/(2q)^{1/2} - Q/2\,(2q - Q^2)^{1/2}$$

7.8 Given that f_1, f_2, and g be any three functions of the canonical variables q_1, ..., q_n, p_1, ..., p_n and of the time t, show that

$$\{f_1 f_2, g\} = f_1 \{f_2, g\} + \{f_1, g\} f_2$$

7.9 Determine the Poisson brackets formed from the cartesian components of the linear momentum $\vec{p}$ and the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ of a particle.

7.10 Determine the Poisson brackets formed from the components of $\vec{L}$.

7.11 *a*) Show that if the Hamiltonian H and the quantity F are constants of motion, then $\partial F/\partial t$ must also be a constant of motion.

b) As an illustration of this result, consider the uniform motion of a free particle of mass m. The Hamiltonian is certainly conserved, and there exists a constant of motino $F = x - pt/m$. Show by direct computation that $\partial F/\partial t$, which is a constant of motion, agrees with $\{H, F\}$.

SOLUTIONS

7.1 Since the transformation equations are independent of time, therefore, $K = H$. For the transformation to be canonical

$$(p\dot{q} - H) - (P\dot{Q} - K) = p\dot{q} - P\dot{Q}$$

must be perfect differential of some function F, that is, must be equal to dF/dt. Effectively

$$pdq - PdQ = dF$$

Now

$$pdq - PdQ = pdq - q \cot p \times \frac{1}{1/q \sin p} \times \frac{q \cos p dp - \sin p dq}{q^2}$$

$$= pdq - q \cot^2 p\, dp + \cot p dq$$

Adding and subtracting qdp from the r.h.s. of the above equation, we get

$$pdq - PdQ = d(pq) - q \operatorname{cosec}^2 p\, dp + \cot p dq$$

$$= d(pq) + d(q \cot p)$$

$$= d(pq + q \cot p)$$

which is perfect differential of $pq + q \cot p$. Therefore, the given transformation is canonical.

7.2 Since the transformation equations are independent of time, therefore, $K = H$. In the present case

$$pdq - PdQ = pdq - \tfrac{1}{2}(p^2 + q^2)(pdq - qdp)/(q^2 + p^2) \qquad (61)$$

as $\tan Q = q/p$

or $\sec^2 Q\, dQ = (pdq - qdp)/p^2$

or $(1 + \tan^2 Q)\, dQ = (pdq - qdp)/p^2$

or $(1 + q^2/p^2)\, dQ = (pdq - qdp)/p^2$

or $dQ = (pdq - qdp)/(p^2 + q^2)$

From equ. (61),

$$pdq - PdQ = pdq - \tfrac{1}{2} pdq + \tfrac{1}{2} q\, dp = d(\tfrac{1}{2} pq)$$

which is perfect differential of ½ pq. Therefore, the transformation is canonical

7.3 Since the transformation equations are independent of time, therefore, $K = H$. In the present case

$$pdq - PdQ = pdq - qp^2(-1/p^2)\, dp = d(pq)$$

which is perfect differential of pq. Therefore, the transformation is canonical.

7.4 Given

$$F_3 = \sum_i p_i Q_i$$

The transformation equations are

$$q_i = -\frac{\partial F_3}{\partial p_i} = -Q_i,$$

$$P_i = -\frac{\partial F_3}{\partial Q_i} = -p_i,$$

Therefore F_3 generates reflection transformation.

7.5 Since the transformation equations are independent of time, therefore, $K = H$. Further in the present case

$$pdq - PdQ = pdq - 2(1 + q^{1/2}\cos p)\, q^{1/2}\sin p \left[\frac{1}{1 + q^{1/2}\cos p}\right.$$

$$\left.(-q^{1/2}\sin p\, dp - \cos p \times \tfrac{1}{2} \times q^{-1/2}\, dq)\right]$$

$$= pdq + q(1 - \cos 2p)\, dp - \tfrac{1}{2}\sin 2p\, dq$$

$$= d(pq - \tfrac{1}{2} q \sin 2p) = dF$$

which is a perfect differential (of F) and therefore, the transformation is canonical.

The above generating function, $F = pq - \tfrac{1}{2}q\sin 2p$ is truly speaking of $F_1(q, Q)$ type as p can be written as function of q, Q from the transformation relation,

$Q = \log(1 + q^{1/2}\cos p)$. Since we are interested in finding $F_3(p, Q)$, using equ. (15), we get

$$\begin{aligned} F_3(p, Q) &= F_1(q, Q) - pq \\ &= pq - \tfrac{1}{2}q\sin 2p - pq \\ &= -q\sin p\cos p \qquad (62) \end{aligned}$$

where the value of $F = F_1$ has been put from above.

Now

$$Q = \log(1 + q^{1/2}\cos p)$$

or $$e^Q = 1 + q^{1/2}\cos p$$

or $$q = (e^Q - 1)^2/\cos^2 p$$

Putting the value of q from above into equ. (62), we get

$$F_3 = -(e^Q - 1)^2\tan p.$$

7.6 Since the transformation equations do not contain time, therefore, $K = H$. For the transformation to be canonical $pdq - PdQ$ must be equal to perfect differential of some function, say F. Therefore, in the present case,

$$\begin{aligned} pdq - PdQ &= pdq - q^{\alpha}\sin\beta p[q^{\alpha}\,\beta(-\sin\beta p)\,dp + \cos\beta p\,\alpha\, q^{\alpha-1}dq] \\ &= (p - \alpha/2\, q^{2\alpha-1}\sin 2\beta p)\,dq + q^{2\alpha}\beta\sin^2\beta p\,dp \\ &= dF \end{aligned}$$

F being function of p and q (as p can be written as a function of q and Q, F amounts to be a function of q, Q) is nothing but F_1. Therefore,

$$(p - \alpha/2\, q^{2\alpha-1}\sin 2\beta p)\,dq + q^{2\alpha}\beta\sin^2\beta p\,dp = dF_1 \qquad (63)$$

$$= \frac{\partial F_1}{\partial q}dq + \frac{\partial F_1}{\partial p}dp \qquad (64)$$

Comparing both the sides, we get

$$\frac{\partial F_1}{\partial q} = p - \alpha/2\, q^{2\alpha-1}\sin 2\beta p, \qquad (65)$$

$$\frac{\partial F_1}{\partial p} = \beta q^{2\alpha}\sin^2\beta p \qquad (66)$$

Differentiating (65) w.r.t. p and (66) w.r.t. q and then equating the two resulting equations, we get

$$1 - \alpha\beta\, q^{2\alpha-1}\cos 2\beta p = 2\alpha\beta\, q^{2\alpha-1}\sin^2\beta p$$

or $\alpha\beta\, q^{2\alpha-1}(2\sin^2\beta p + 1 - 2\sin^2\beta p) = 1$

or $\alpha\beta\, q^{2\alpha-1} = 1$

Since on the r.h.s., the power of q is zero, therefore

$$2\alpha - 1 = 0 \text{ or } \alpha = ½$$

and hence

$$\beta = 2$$

Putting the values of α and β in equ. (63), we get

$$\begin{aligned} dF_1 &= (p - ¼ \sin 4\,p)\,dq + 2\,q \sin^2 2p\,dp \\ &= pdq - ¼ \sin 4\,p\,dq + q(1 - \cos 4p)\,dp \\ &= (pdq + qdp) - (¼ \sin 4\,p\,dq + q \cos 4p\,dp) \\ &= d\,(pq - q/4 \sin 4p) \end{aligned}$$

which amounts to the fact that

$$F_1 = pq - q/4 \sin 4p \qquad (67)$$

To find the generating function F_3, we use the relation

$$\begin{aligned} F_3 &= F_1 - pq \\ &= -q/4 \sin 4p \end{aligned}$$

where the value of F_1 has been put from equ. (67).

Now since

$$Q = q^{1/2} \cos 2p,$$

therefore,

$$q = Q^2 \sec^2 2p$$

As a result,

$$\begin{aligned} F_3 &= -Q^2/4 \sec^2 2p \sin 4\,p \\ &= -Q^2/4 \sec^2 2p \times 2 \sin 2p \cos 2p \\ &= -½\,Q^2 \tan 2p \end{aligned}$$

7.7 In the present case since the transformation equations do not contain time, therefore, $K = H$. Now

$$\begin{aligned} pdq - PdQ &= pdq - (2q)^{1/2} \sin p\{(2q)^{1/2} \times -\sin p\,dp + ½ \\ &\qquad \times (2q)^{-1/2}\,dq \cos p\} \\ &= pdq + 2q \sin^2 p\,dp - q \cos p \sin pdq \\ &= (p - q \cos p \sin p)\,dq + 2\,q \sin^2 p\,dp \\ &= d(qp - q \sin p \cos p) \end{aligned}$$

which is a perfect differential of function

$$F = F_1 = q(p - \sin p \cos p)$$

and hence the transformation is canonical.

From the transformation equations

$$p = \cos^{-1} Q/(2q)^{1/2}$$

and

$$Q^2 = 2q\cos^2 p = 2q\,(1 - \sin^2 p)$$

or $$\sin p = (2q - Q^2)^{1/2}/(2q)^{1/2}$$

Therefore,

$$q \sin p \cos p = q\,(2q - Q^2)^{1/2}/(2q)^{1/2} \times Q/(2q)^{1/2}$$
$$= Q/2\,(2q - Q^2)^{1/2}$$

As a result, the generating function

$$F = F_1(q, Q) = q\cos^{-1} Q/(2q)^{1/2} - Q/2\,(2q - Q^2)^{1/2}$$

7.8 From the definition of Poisson's bracket

$$\{f_1 f_2, g\} = \sum_i \left(\frac{\partial(f_1 f_2)}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial(f_1 f_2)}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

$$= \sum_i \left\{\left(\frac{\partial f_1}{\partial q_i} f_2 + f_1\frac{\partial f_2}{\partial q_i}\right)\frac{\partial g}{\partial p_i} - \left(\frac{\partial f_1}{\partial p_i} f_2 + f_1\frac{\partial f_2}{\partial p_i}\right)\frac{\partial g}{\partial q_i}\right\}$$

$$= \sum_i \left[f_1\left(\frac{\partial f_2}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f_2}{\partial p_i}\frac{\partial g}{\partial q_i}\right) + f_2\left(\frac{\partial f_1}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f_1}{\partial p_i}\frac{\partial g}{\partial q_i}\right)\right]$$

$$= f_1\{f_2, g\} + f_2\{f_1, g\} \tag{68}$$

Therefore, one can conclude that

$$\{f, g_1 g_2\} = -\{g_1 g_2, f\}$$
$$= -[g_1\{g_2, f\} + g_2\{g_1, f\}]$$
$$= g_1\{f, g_2\} + g_2\{f, g_1\} \tag{69}$$

7.9 Putting $g = q_i$ in equ. (32), one gets

$$\{\mathrm{f}, \mathrm{q}_i\} = -\frac{\partial f}{\partial p_i} \tag{70}$$

As the sum reduces to a single term because of the fact that $\partial q_i/\partial q_l = \delta_{il}$ and $\partial q_i/\partial p_l = 0$. Similarly putting $g = p_i$ in equ. (32), one gets

$$\{f, p_i\} = \partial f/\partial q_i \tag{71}$$

From equ. (71) above,

$$\{L_x, p_y\} = \frac{\partial L_x}{\partial y} = \partial/\partial y\,(yp_z - z\,p_y) = p_z$$

$$\{L_x, p_x\} = \frac{\partial L_x}{\partial x} = 0$$

$$\{L_x, p_z\} = \frac{\partial L_x}{\partial z} = \partial/\partial z\,(yp_z - zp_y) = -p_y$$

The remaining Poisson brackets are obtained by cyclically permuting the suffixes x, y, z

7.10 $\{L_x, L_y) = \{y\, p_z - z\, p_y , L_y\}$

$$= y\, \{p_z, L_y\} + \{y, L_y\}\, p_z - z\{p_y, L_y\} - \{z, L_y\}\, p_y$$

where we have used the relation,

$$\{f_1 f_2, g\} = f_1\, \{f_2, g\}\, \{f_1, g\} f_2$$

Further

$$\{y, L_y\} = \{p_y, L_y\} = 0\,,$$

therefore

$$\begin{aligned}\{L_x, L_y\} &= y\, \{p_z, L_y\} - \{z, L_y\}\, p_y \\ &= y\left(-\frac{\partial L_y}{\partial z}\right) - \left(\frac{\partial L_y}{\partial pz}\right) p_y \\ &= -y\, \partial/\partial z\, (z\, p_x - x\, p_z) - p_y\, \partial/\partial p_z\, (zp_x - xp_z) \\ &= -yp_x + xp_y \\ &= L_\mathrm{z}\end{aligned}$$

Similarly, we can show that

$$\{L_y, L_z\} = L_x,\ \ \{L_z, L_x\} = L_y$$

7.11 (a) The total time derivative of F is given by

$$dF/dt = \partial F/\partial t + \{F, H\}$$

From poisson theorem, we known that if F and H are two coustants of motion, then their poission bracket $\{F, H\}$ world also be a constant of motion. Further since F is a constant of motion

$$dF/dt = 0$$

and as a result

$$0 = \partial F/\, \partial t + \text{const. of motion}$$

implying that $\partial F/\, \partial t$ world also be a cosstant of motion.

(b) Given

$$F = x - pt/m = q - pt/m \tag{72}$$

therefore, $$\partial F/\partial t = -\,p/m \tag{73}$$

Now $$\{F, H\} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} \tag{74}$$

In this case,

$$H = p^2/2m + 0 \tag{75}$$

as the particle is free and hence $V = 0$

Further from equs. (75) and (72), we have

$$\frac{\partial H}{\partial q} = 0, \ \frac{\partial H}{\partial p} = p/m, \ \frac{\partial F}{\partial q} = 1$$

Putting above into (74), we get

$$\{H, F\} = -\{F, H\} = -p/m = \partial F/\partial t$$

in which equ. (73) has been used.

8
CHAPTER

Hamilton - Jacobi Theory

8.1 HAMILTON-JACOBI EQUATION

The equations of motion in terms of the transformed (or new) Hamiltonian K and the relation between the new and the old Hamiltonian H are

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i, -\frac{\partial K}{\partial Q_i} = \dot{P}_i, K(Q, P, t) = H(q, p, t) + \frac{\partial F}{\partial t} \quad (1)$$

where F is the generating function of the transformation. If the new variables Q_i and P_i are constants of motion, then the equations of motion are

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0\,, -\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0 \quad (2)$$

In other words if we require that the transformed Hamiltonian $K = 0$ we get the above equations of motion which obviously insure that the new variables Q_i, P_i are constants of motion. The generating function F then satisfies the equation

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0 \quad (3)$$

Let us choose F as a function of the old coordinates q_i, and the new constant momenta P_i, and the time or in other words we designate the generating function F as $F_2(q, P, t)$. Further using the equations of transformation,

$$p_i = \frac{\partial F_2}{\partial q_i}\,,$$

equ (3) becomes

$$H(q_1, \ldots\ldots\ldots, q_n; \frac{\partial F_2}{\partial q_1}, \ldots\ldots\ldots\ldots, \frac{\partial F_2}{\partial q_n}; t) + \frac{\partial F_2}{\partial t} = 0 \quad (4)$$

The above first order partial differential equation in $(n + 1)$ variables, $q_1, \ldots\ldots\ldots\ldots, q_n$; t is known as the Hamilton Jacobi equation for the desired generating function. It is customary to denote the solution of equ. (4) by S (instead of F_2) called Hamilton's principal function.

Since equ. (4) is a partial differential equation in $(n + 1)$ variables, its complete solution must contain $(n + 1)$ independent arbitrary constants of integration say

α_1, ... α_{n+1}. However, since S does not appear directly in equ. (4), but only the partial derivatives of S w.r.t. q or t appear in the said equation, one of the constants α_1.........α_{n+1}, say α_{n+1} is purely additive and disappear on differentiation. That is, if S is solution of equ. (4), then $S + \alpha$ must also be the solution where α is ay additive constant. As a result the differential of α would be zero and that of S would remain. Therefore, the complete solution of equ. (4) can be written in the form

$$F_2 = S = S(q_1, q_2, ..., q_n; \alpha_1, ..., \alpha_n; t) \tag{5}$$

that is, S is a function of n coordinates q_i, the time t, and n independent constants α_i. This is precisely the same description as that of the F_2 type of generating function. We can, therefore, take the n constants of integration to be the new (constant) momenta. Thus

$$P_i = \alpha_i \quad (i = 1, ..., n) \tag{6}$$

The n transformation equations can now be written as

$$p_i = \frac{\partial S(q_i, \alpha_i, t)}{\partial q_i} \tag{7}$$

and

$$Q_i = \beta_i, \text{ a const.} = \frac{\partial S(q_i, \alpha_i, t)}{\partial P_i} = \frac{\partial S(q_i, \alpha_i, t)}{\partial \alpha_i} \tag{8}$$

where β_1, ... β_n are new (constant) coordinates.

By means of the n equations (8), the n coordinates can be expressed in terms of the time t and the $2n$ constants α_i, β_i, that is

$$q = q(\alpha_i, \beta_i, t) \tag{9}$$

After the differentiation in equ. (7) has been performed, equ. (9) may be substituted for q's, thus giving n momenta as functions of α_i, β_i and time t, that is

$$p = p(\alpha_i, \beta_i, t) \tag{10}$$

Equs. (9) and (10) thus constitute the desired complete solution of the problem.

To have insight into the physical significance of the Hamilton's principal function.

$$S = S(q_i, \alpha_i, t),$$

we examine its total time derivative,

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial S}{\partial \alpha_i}\dot{\alpha}_i = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i$$

as $\dot{P}_i = \dot{\alpha}_i = 0$

Further using equs. (7) and (4), the above equation takes the form

$$\frac{dS}{dt} = \sum_i p_i \dot{q}_i - H = L \tag{11}$$

Therefore,

$$S = \int L dt + \text{const.} \tag{12}$$

showing that Hamilton's principal function differs from the indefinite time integral of L by at most a constant. This furnishes another (one may recall that Hamilton's principle is a statement about the definite integral of L from which we obtained the solution of the problem via Lagrange equations) way of solving the problem. In actual calculations the result expressed by equ. (12) is of no help because one can not integrate L w.r.t. time until q_i and p_i are known as functions of time, that is, until the problem is solved.

8.2 METHODS FOR SOLVING THE HAMILTON - JACOBI EQUATION

Truly speaking there is no general method of finding complete solution of the Hamilton-Jacobi equation. However, in some particular cases, a complete solution can be found out by separation of variables.

(i) Separation of Time

If the Hamiltonian H does not involve time explicitly, we look for a solution of the Hamilton - Jacobi equation of the form

$$S(q_i, \alpha_i, t) = W(q_i, \alpha_i) + S_1(\alpha_i, t) \tag{13}$$

Using above, equ. (4) becomes

$$\partial S_1(\alpha_i, t)/\partial t + H(q_i, \partial W(q_i, \alpha_i)/\partial q_i) = 0 \tag{14}$$

Since the first term depends only on the time variable, whereas the second on the independent variables q_i, equ. (14) is satisfied only if the first term equals a constant $-E$ and the second one equals E. Thus

$$\frac{\partial S_1}{\partial t} = -E, \tag{15}$$

which on integration gives

$$S_1(\alpha_i, t) = -Et \tag{16}$$

where we have omitted the additive constant. We also have

$$H(q_1, \ldots\ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots\ldots\ldots \frac{\partial W}{\partial n_1}) = E \tag{17}$$

called the restricted (or abbreviated) Hamilton - Jacobi equation. Since the Hamiltonian does not involve time explicitly, E is usually the total energy of the system, but it should be remembered that this need not always be the case see Sec. 6.3.

Since the number of independent constants appearing in S must be n, it is customary to choose the constant E as α_1. Then the complete solution (13) reads:

$$S(q_1, \ldots\ldots, q_n, \alpha_1, \ldots\ldots, \alpha_n, t) = -Et + W(q_1, \ldots\ldots, q_n, E, \alpha_2, \ldots\ldots, \alpha_n) \tag{18}$$

The time independent function W appearing here as a part of the generating function S when H is constant, is called Hamilton's characteristic function. We would now show that W separately generates its own canonical transformation with properties quite different from that generated by S. Consider a canonical transformation in which the new momenta P_i are all constants of motion α_i, and where α_1 is equal to H. If the generating function for this transformation be denoted by $W(q, P)$, then the equations of transformation are

$$p_i = \frac{\partial W}{\partial q_i}, Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i} \tag{19}$$

While the above equations resemble equs. (7) and (8) for Hamilton's principal function S, the condition now determining W is that H shall be equal to the new momentum α_1:

$$H(q_i, p_i) = \alpha_1 \tag{20}$$

With the substitution of p_i from equ. (19) into above, one gets

$$H(q_i, \frac{\partial W}{\partial q_i}) = \alpha_1 \tag{21}$$

which is seen to be identifical with equ. (17). Further since W does not involve the time, the new Hamiltonian

$$K = H + \frac{\partial W}{\partial t} = H + 0 = \alpha_1 \tag{22}$$

Since the new coordinates Q_i do not appear in K (see equ. 21), therefore are cyclic. Thus the function W generated a canonical transformation in which all the new coordinates are cyclic. The canonical equations for P_i merely repeated the statement that the momenta conjugate to the cyclic coordinates are all constant:

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0, P_i = \alpha_i \tag{23}$$

Because the new Hamiltonian K depends on only one of the momenta α_1, the equations of motion for Q_i are

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = \frac{\partial K}{\partial \alpha_i} = \frac{\partial \alpha_1}{\partial \alpha_i} = 1 \text{ for } i = 1$$
$$= 0 \text{ for } i \neq 1$$

with the immediate solutions

$$Q_1 = t + \beta_1 = \frac{\partial W}{\partial \alpha_1},$$
$$Q_i = \beta_i = \frac{\partial W}{\partial \alpha_1}, \quad i \neq 1 \tag{24}$$

Thus, the only coordinate which is not a constant of the motion is Q_1, which is equal to the time plus a constant. We have here an example of time as a coordinate and Hamiltonian as its conjugate momentum.

The Hamilton's characteristic function $W(q_i, \alpha_i)$ has physical significance similar to that for S. The total time derivative of W is

$$\frac{dW}{dt} = \sum_i \frac{\partial W}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial W}{\partial \alpha_i}\dot{\alpha}_i = \sum_i p_i \dot{q}_i$$

where we have used the facts that $\dot{\alpha}_i = 0$ and $p_i = \dfrac{\partial W}{\partial q_i}$. Therefore,

$$W = \int \sum_i p_i \, dq_i \tag{25}$$

which is recognized as defining the abbreviated action

(ii) Separation of Variables

It is possible to separate the variables in the Hamilton-Jacobi equation under certain conditions and the solution can then be reduced to quadratures. Truly speaking the Hamilton-Jacobi formulation becomes a useful tool only when such a separation can be affected.

A particular case of separable variables is the one in which the variables are cyclic. Assuming that the variable q_1 is taken as a separable coordinate, then we look for the solution S of the form.

$$S(q_1, .. , q_n; \alpha_1, ... \alpha_n; t) = S_1(q_1; \alpha_1, ... \alpha_n; t) + S'(q_2, .. q_n; \alpha_1, ... \alpha_n; t) \tag{26}$$

and that effectively means that the Hamilton-Jacobi equation can be split into two equations, one separately for S_1 and the other for S'. Similarly the Hamilton - Jacobi equation is described as completely separable if all the coordinates in the problem are separable. A solution for Hamilton's principal function of the form

$$S = \sum_i S_i(q_i, \alpha_1, ..., \alpha_n; t) \tag{27}$$

would then split the Hamilton-Jacobi equation into n equations of the type

$$H_i\left(q_i, \frac{\partial S_i}{\partial q_i}, \alpha_1, ..., \alpha_n\right) = \alpha_i, \tag{28}$$

the constants α_i here are referred to as the separation constants. Each of the equs. (28) involves only one of the coordinates q_i(Note that there is no summation in equ. 28) and the corresponding partial derivative of S_i w.r.t. q_i. These are, therefore, set of n ordinary differential equations of a particularly simple form. Since the equations are only of the first order, it is always possible to reduce them to quadratures. One has only to solve for the partial derivatives of S_i w.r.t. q_i and integrate over q_i.

If suppose all except one coordinate say q_1 are cyclic, the separated form for W then appears as†

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† The form (29) is arrived at by the consideration that W is to be the generating function for a transformation to new coordinates that are all cyclic. But if q_2, ..., q_n are already cyclic, no further transformation is needed for them. As far as they are concerned W can be identity transformation since α_i's are the new momenta, the summation in (29) can be written as $\sum_{i=2}^{n} q_i P_i$. which would be recognized as the generator of the identify transformation for the coordinates q_2, ..., q_n.

$$W = \sum_{i=1}^{n} W_i(q_i, \alpha) = W_1(q_1, \alpha) + \sum_{i=2}^{n} \alpha_i q_i \tag{29}$$

where W_1 corresponds to $i = 1$. Then the Hamilton-Jacobi equation reduces to

$$H\left(q_1, \frac{\partial W_1}{\partial q_1}, \alpha_2, \ldots, \alpha_n\right) = \alpha_1 \tag{30}$$

which is an ordinary first order differential equation in the independent variable q_1 and hence can be solved immediately.

iii) Separation of Pair Variables

Let us assume that some coordinate q_j say, and the conjugate momentum $p_j = \partial W/\partial q_j$ appear in the Hamiltonian - Jacobi equation only as the combination

$$f\left(q_j, \frac{\partial W}{\partial q_j}\right) \tag{31}$$

that does not involve any of other variables. If we then seek a trial solution of the form.

$$W = W_j(q_j, \alpha) + W'(q_i, \alpha)$$

where q_i represents the set of all q's except q_j. The Hamilton-Jacobi equation then takes the form

$$H\left(q_i, \frac{\partial W'}{\partial q_i}, f\left(q_j, \frac{\partial W_j}{\partial q_j}\right)\right) = \alpha_1 \tag{33}$$

In principle, the above equation can be inverted so as to solve for f:

$$f\left(q_j, \frac{\partial W_j}{\partial q_j}\right) = g\left(q_i, \frac{\partial W'}{\partial q_i}, \alpha_1\right) \tag{33}$$

Note that f is not a function of any of the q's except q_j, g on the other hand is independent of q_j. Therefore, the above equation can hold good only if both sides are equal to the same separation constant, independent of all q's, that is

$$f\left(q_j, \frac{\partial W_j}{\partial q_j}\right) = \alpha_j,$$

$$g\left(q_i, \frac{\partial W'}{\partial q_i}, \alpha_1\right) = \alpha_j, \tag{34}$$

where α_j is an arbitrary constant and the separation of the variables has been accomplished.

8.3 APPLICATIONS

Before applying the Hamilton-Jacobi technique to the following problems it is necessary to mention that the separability of the Hamilton-Jacobi equation depends not only on the physical problem involved but also on the choice of the generalized coordinates to be used. For example, the one body central force problem is separable in polar coordinates but not in the cortesion coordinates. In the case of the three body problem, it is not at all possible to completely separate the Hamilton-Jacobi equation. In many of the basic problems in mechanics separation is possible in more than one set of coordinates. In general, it may be pointed out that it is feasible to solve the Hamilton-Jacobi equation in closed form only when the variables are completely separable.

(i) Linear Harmonic Oscillator

Since the system has one degree of freedom and the Hamiltonian.

$$H = p^2/2m + \tfrac{1}{2} k x^2, \omega = (k/m)^{1/2} \tag{35}$$

in the present case being independent of time, we search for a complete solution of the form

$$S = W(x, E) - Et \tag{36}$$

The restricted Hamilton - Jacobi equation,

$$1/2m \left(\frac{dW}{dx}\right)^2 + \tfrac{1}{2} kx^2 = E \tag{37}$$

has the solution (or on integration yields)

$$W(x, E) = \int (2mE - m^2\omega^2x^2)^{1/2}\, dx + C$$

where C is a constant of integration. Thus

$$S = \int (2mE - m^2\omega^2x^2)^{1/2}\, dx - Et + C \tag{38}$$

where use of equ. (36) has been made.

The solution for x arises out of the transformation equ. (8),

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{\partial S}{\partial E} = (2m/E)^{1/2} \int \frac{dx}{\left(1 - \frac{m\omega^2 x^2}{2E}\right)^{\frac{1}{2}}} - t$$

or $$t + \beta = 1/\omega \sin^{-1} x/(2E/m\omega^2)^{1/2}$$

or $$x = (2E/m\omega^2)^{1/2} \sin(\omega t + \theta) \tag{39}$$

where we have made the substitution $\theta = \beta\omega$. Equ. (39) is the familiar solution for a linear harmonic oscillator.

Formally, the solution for the momentum comes from the transformation equ. (7), which using equ. (38) can be written as

$$p = \frac{\partial S}{\partial x} = (2mE - m^2\omega^2x^2)^{1/2}$$

$$= (2mE)^{1/2} \cos(\omega t + \theta) \tag{40}$$

where we have substituted for x from equ. (39), Equ. (40) is the expected result and it checks with the identification of p as $m\dot{x}$.

(ii) Central Force (or Kepler's) Problem

It is well known that the path of a particle subject to such a force lies in a plane. Therefore, we choose $q_1 = r$, $q_2 = \theta$ as the generalized coordinates for the problem under consideration. The kinetic and potential energies of the particle are given by

$$T = ½\, m(\dot{r}^2 + r^2\dot{\theta}^2),$$

and $$V = -k/r$$

Therefore, the Lagrangian of the particle is

$$L = T - V = ½\, m(\dot{r}^2 + r^2\dot{\theta}^2) + k/r$$

and hence

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, p_\theta = \frac{\partial L}{\partial \theta} = mr^2\dot{\theta} \tag{41}$$

The Hamiltonian of the particle is, then

$$H = T + V = 1/2m\,(\,p_r^2 + p_\theta^2/r^2) - k/r \tag{42}$$

The Hamiltonian being cyclic in θ, the Hamilton's characteristic function can be written as (see equ. 29)

$$W = W_1\,(r) + \alpha_\theta\theta \tag{43}$$

where α_θ is the constant angular momentum p_θ conjugate to θ. The Hamilton-Jacobi equation then becomes

$$1/2m\left[\left(\frac{\partial W_1}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2}\right] - k/r = \alpha_1 = E \tag{44}$$

where α_1 is the constant identified physically as the total energy E of the system. Solving equ. (44) for the partial derivative of W_1, we obtain.

$$\frac{\partial W_1}{\partial r} = \sqrt{2m(E + k/r) - \alpha_\theta/r^2}$$

Therefore,

$$W = W_1 + \alpha_\theta\,\theta = \int\sqrt{2m(E + k/r) - \alpha_\theta^2/r^2}\;dr + \alpha_\theta\,\theta$$

plus a constant of integration which would finally lead to zero.

With the above form of the Hamilton's characteristic function, the transformation equs. (24) yield

$$t + \beta 1 = \frac{\partial W}{\partial \alpha_1} = \frac{\partial W}{\partial E} = \int \frac{m\,dr}{[2m(E + k/r) - \alpha_\theta^2/r^2]^{1/2}} \tag{45}$$

and
$$\beta_2 = \frac{\partial W}{\partial \alpha_\theta} = -\int \frac{\alpha_\theta dr}{r^2[2m(E+k/r)-\alpha_\theta^2/r^2]^{1/2}} + \theta \qquad (46)$$

Equ. (45) furnishes r as a function of t and agrees with equ. (19) of chapter 3 with α_θ written as l whereas equ. (46) provides orbit equation as it relates r with θ. If the variable of integration in (46) is changed to $u = 1/r$, the equation reduces to

$$\theta = \beta_2 - \int \frac{du}{[2m/\alpha_\theta^2\,(E+ku)-u^2]^{1/2}} \qquad (47)$$

which agree with equ. (32) of chapter 3 found for the orbit, identifying β_2 as θ' and of course α_θ as l.

From equ. (47),

$$\beta_2 - \theta = \int \frac{du}{[2m/\alpha_\theta^2 + m^2k^2/\alpha_\theta^4 - (u - mk/\alpha_\theta^2)^2]^{1/2}}$$

$$= \sin^{-1}\left[(u - mk/\alpha_\theta^2)/(2mE/\alpha_\theta^2 + m^2k^2/\alpha_\theta^4)^{1/2}\right]$$

or
$$u = 1/r = \frac{mk}{\alpha_\theta^2} + \left(\frac{2mE}{\alpha_\theta^2} + \frac{m^2k^2}{\alpha_\theta^4}\right)^{1/2} \sin(\beta_2 - \theta)$$

$$= \frac{mk}{\alpha_\theta^2}\left[1 + \left(1 + \frac{2E\alpha_\theta^2}{mk^2}\right)^{1/2} \cos(\theta - \theta')\right],$$

$$\beta_2 = \pi/2 + \theta'$$

Using the notation

$$1/l = \frac{mk}{\alpha_\theta^2}, \in = \left(1 + \frac{2E\alpha_\theta^2}{mk^2}\right)^{1/2}$$

we get

$$1/r = 1/l\,[1 + \in \cos(\theta - \theta')] \qquad (48)$$

which implies that a particle subjected to a central force field describes a conic.

8.4 ACTION - ANGLE VARIABLES

When the motion of a physical system is periodic[††] in nature, an elegant and powerful method of studying these problems derives from the Hamilton-Jacobi formalism. In this technique, the integration constants α_i appearing in the solution of the Hamilton-Jacobi equation are not themselves chosen to be the new momenta, instead, we use suitably defined constants J_i, which form a set of n independent functions of the α_i's and are called action variables. The said technique was historically important in the development of quantum theory.

[††] Two types of periodic motion may be distinguished. In the first type the orbit or trajectory of the system point in phase space is a closed curve and the motion is called vibration or libration. In this case both q and p are periodic functions of time with the same period. The linear harmonic oscillator falls in this category. In the second type of periodic motion the orbit in the phase space is such that p is periodic function of q with period q_0. The periodic motion in this case is termed as rotation or revolution. When q is increased by q_0, the configuration of the system remains essentially unchanged. The most familiar example of this type of periodic motion is that of a rigid body constrained to rotate about a fixed axis with q as the angle of rotation. Increasing q by 2π then produces no essential change in the state of the system.

For either type of periodic motion (vibration or rotation), we define a new variable J_i called action variable as

$$J_i = \oint p_i \, dq_i \tag{49}$$

where the integration is to be carried over a complete cycle or period of the coordinate q_i indicated by the circle on the integral sign, that is over the complete period of vibration or of rotation, as the case may be. From the transformation equs. (19),

$$p_i = \frac{\partial W}{\partial q_i} = \partial W_i(q_i, \alpha_1, ..., \alpha_n) / \partial q_i$$

and therefore,

$$J_i = \oint \partial W_i(q_i, \alpha_1, ..., \alpha_n) \partial q_i \; dq_i$$

Note that the integration in the above equation leaves J_i as a function of the α's only. Therefore,

$$J_i = J_i(\alpha_1, ..., \alpha_n) \tag{50}$$

It is obvious that J_i has the dimensions of angular momentum. The above n relations could be inverted to yield

$$\alpha_i = \alpha_i(J_1, ... , J_n) \tag{51}$$

By means of the above relations, the Hamilton's characteristic function W can be written as

$$W = W(q_1, ..., q_n, J_1, ..., J_n) \tag{52}$$

The Hamiltonian now becomes (as W is generator of type 2)

$$K = H = \alpha_1 = \alpha_1(J_1, ..., J_n) = H(J_1, ... J_n) \tag{53}$$

where we have made use of equ. (51)

Note that the J_i themselves are chosen as the new momenta $P_i = J_i$ (see equ. 52). The generalized coordinates conjugate to J_i are called the angle variables ω_i and are defined by the transformation equations

$$Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial J_i} = \omega_i \tag{54}$$

Since the coordinates ω_i are conjugate to angular momenta, their dimensions are those of angles.

The new canonical equations of motion are

$$\dot{P}_i = \dot{J}_i = -\frac{\partial K}{\partial Q_i} = -\frac{\partial K}{\partial \omega_i} = 0$$

as K is a function of J's only. The above equation implies that J's are constant (which is already known from equs. 50). And

$$\dot{Q}_i = \dot{\omega}_i = \partial K/\partial P_i = \partial H/\partial P_i = \partial H(J_1, \ldots\ldots\ldots\, J_n)\,/\partial J_i$$
$$= \nu_i(J_1, \ldots\ldots, J_n) \tag{55}$$

Integrating above equation, we obtain

$$\omega_1 = \nu_i\, t + \tau_i \tag{56}$$

where τ_i is the integration constant. Each ν_i of course depends on the constants J_r. Thus ω_i is a linear function of time.

To find the physical significance of ν's, let us determine the variation of the angle variable ω_i when one of the coordinates q_j goes through a full period of motion while the others are held fixed. We have

$$\Delta\omega_i = \oint \frac{\partial \omega_i}{\partial q_j} dq_j = \oint \frac{\partial}{\partial q_j}\left(\frac{\partial W}{\partial J_i}\right) dq_j$$

$$= \frac{\partial}{\partial J_i} \oint \frac{\partial W}{\partial q_j} dq_j$$

$$= \frac{\partial}{\partial J_i} \oint p_j\, dq_j$$

where we have made use of equs. (54) and the transformation equ. (19). Further using equ. (49), we get

$$\Delta\omega_i = \frac{\partial J_j}{\partial J_i} = \delta_{ij} = 1 \text{ for } i = j$$
$$= 0 \text{ for } i \neq j \tag{57}$$

The above equation shows that ω_i changes by unity when $q_j = q_i$ goes through a complete cycle. Further using equ. (56),

$$\Delta\omega_i = \nu_i\, \Delta t = \nu_i\, \tau_i = 1$$

or
$$\nu_i = 1/\tau_i \tag{58}$$

where $\Delta t = \tau_i$ is the period for a complete cycle of q_i. Therefore, ν_i is evidently the frequency of this periodic motion. The use of action-angle method thus provides a very direct procedure for finding all the allowed frequencies of the periodic motion. First, the action integrals are worked out as a function of $\alpha_i(J)$ as in equ. (51). Finally, the Hamiltonian $H(J)$ is differentiated with respect to J_i by equ. (55), thus yielding the frequencies ν_i. The procedure is, therefore, both elegant and straight forward.

8.5 APPLICATIONS

As illustrations of the applications of action-angle variables method, let us apply it to the following two problems.

(i) Linear Harmonic Oscillator

From equ. (37) and the defining equ. (49), the action variable J is given by

$$J = \oint p dx = \oint \sqrt{2mE - m^2\omega^2 x^2}\, dx$$

as the case is one-dimensional and also that the Hamiltonian is not an explicit function of time and that $\omega^2 = k/m$

The substitution

$$x = (2E/m\omega^2)^{1/2} \sin\theta$$

reduces the integral to

$$J = E/\omega\ (\theta + ½ \sin 2\theta)\Big|_0^{2\pi} = 2\pi\ E/\omega \tag{59}$$

and hence

$$E \equiv H = J\omega/2\pi \tag{60}$$

The frequency of oscillation (from equ. 55) is, therefore

$$\nu = \frac{\partial H}{\partial J} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{k/m}$$

which is the formula for the frequency of a linear harmonic oscillator.

(ii) Central Force (orKepler's) problem

For a particle moving in the attractive potential, the Hamiltonian is given by (see equ. 42)

$$\text{H} = 1/2m\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - k/r$$

The Hamiltonian being cyclic in θ, the Hamilton's characteristic function W can be written as

$$W = W_r(r) + \alpha_\theta\, \theta$$

where $\alpha_2 = \alpha_\theta$ is the constant angular momentum p_θ conjugate to θ.

The Hamilton - Jacobi equation then becomes

$$1/2m\left[\left(\frac{\partial W_r}{\partial r}\right) + \frac{\alpha_\theta^2}{r^2}\right] - k/r = \alpha_1 = \alpha_r = E$$

where $\alpha_1 = \alpha_\text{r} = E$ is the total energy of the system. Therefore, the action variables (or integrals) are

$$J_\theta = \oint p_\theta d\theta = \alpha_\theta \oint d\theta = 2\pi\alpha_\theta \tag{61}$$

and $$J_r = \oint p_r \, dr = \oint \frac{\partial W_r}{\partial r} dr = \oint \sqrt{2m(E + k/r) - \alpha_\theta^2 / r^2}\, dr \tag{62}$$

We shall confine our discussion to the bound case, that is, $E < 0$. Hence the motion in each coordinate would be periodic. In this case, since the integrand is equal to $p_r = m\dot{r}$, the limits of motion are defined by the roots $r_{\min}$ and $r_{\max}$ of the expression (equ. 62) in the square root sign. $p_r = m\dot{r}$ is positive as the mass moves from $r_{\min}$ to $r_{\max}$, and is negative as it returns to $r_{\min}$. At these two extreme values of r (turning points on the orbit) $\dot{r} = 0$. Thus $r_{\min}$ and $r_{\max}$ are roots of the equation.

$$2m(E + k/r) - \frac{\alpha_\theta^2}{r^2} = 0$$

or $$r^2 + k/E\, r - \frac{\alpha_\theta^2}{2mE} = 0$$

and are given by

$$\begin{aligned} r &= -k/2E \pm 1/2E\sqrt{k^2 + \frac{2E\alpha_\theta^2}{m}} \\ &= -k/2\text{E}\left(1 \mp \sqrt{1 + \frac{2E\alpha_\theta^2}{mk^2}}\right) \\ &= a\,(1 \pm \in) \end{aligned} \tag{63}$$

where we have used the relations (38) and (41) of chapter 3 for the eccentricity and the semi major axis. Then the integral (62) can be written as

$$J_r = 2\int_{r_{\min}}^{r_{\max}} \sqrt{2m(E + k/r) - \alpha_\theta^2 / r^2}\; dr$$

which we would write as

$$J_r = \oint \sqrt{-A + \frac{2B}{r} - \frac{C}{r^2}}\, dr$$

where

$$A = -2mE,\ B = mk, \text{ and } C = \alpha_\theta^2$$

The above integral can be evaluated using the complex variable theory and the outcome is

$$\begin{aligned} J_r &= 2\pi\left(B/\sqrt{A} - \sqrt{C}\right) \\ &= 2\pi\left(\frac{mk}{\sqrt{-2mE}} - \alpha_\theta\right) \end{aligned} \tag{64}$$

Therefore,

$$J_r + J_\theta = 2\pi mk/\sqrt{-2mE} \tag{65}$$

where we have made use of equs. (61) and (64). As a result

$$\alpha_r = \alpha_1 = H = E = K = -\frac{2\pi^2 mk^2}{(J_r + J_\theta)^2} \tag{66}$$

Then from equ. (55), the frequencies ν_r and ν_θ are given by

$$\nu_r = \frac{\partial H}{\partial J_r} = \frac{4\pi^2 mk^2}{(J_r + J_\theta)^3} = \frac{\partial H}{\partial J_\theta} = \nu_\theta \tag{67}$$

Thus, the only two frequencies involved are equal, a situation known as degeneracy. Therefore the system is degenerate. The period T being reciprocal of the frequency from equs. (67) and (65) is

$$T = 1/\nu_r = 1/\nu_\theta = \frac{(J_r + J_\theta)^3}{4\pi^2 mk^2} = \pi k\sqrt{m/2(-E)^3}$$

or $$T^2 = \frac{\pi^2 k^2 m}{2(-E)^3} = \left(\frac{4\pi^2 m}{k}\right) a^3,$$

where we have used the relation $E = -k/2a$. The above result is nothing else but Kepler's third law.

PROBLEMS

8.1 Set up the problem of the heavy symmetrical top, with one point fixed, in the Hamilton-Jacobi method and obtain the equation of motion of the top.

8.2 Use Hamilton-Jacobi method to describe the motion of a particle falling vertically under the action of the uniform gravitational field.

8.3 Solve the problem of the motion of a point projectile in a vertical plane using the Hamilton-Jacobi method. Find the dependence of the coordinates on time and equation of trajectory, assuming that the projectile is fired off at time $t = 0$ from the origin with the velocity v_0, making an angle θ with the horizontal.

8.4 Using Bohr-Sommerfeld conditions and the action - angle variable theory, obtain energy levels in the case of H-atom.

8.5 Find the frequencies of a three-dimensional harmonic oscillator with unequal force constants using the method of action-angle variables.

8.6 Show by using the method of action-angle variables that if the amplitude of oscillations is small, the energy of a simple pendulum, $E = J\nu$ where terms have their usual meaning.

8.7 Set up the Kepler problem in spherical polar coordinates and show by using the method of action-angle variables that the total energy is given by

$$E = -\frac{2\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^2}$$

where terms have their visual meaning.

8.8 Set up the relativistic Kepler problem (in plane polar coordinates) using the method of action-angle variables and show that the total energy is given by

$$E = m_0 c^2 \Big/ \sqrt{1 + \frac{4\pi^2 k^2}{[cJ_r + (J_\theta^2 c^2 - 4\pi^2 k^2)^{1/2}]^2}}$$

where terms have their usual meaning.

SOLUTIONS

8.1 The Lagrangian for the problem under consideration (see equ. 58- Chapter 4) is given by

$$L = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\phi}\cos\theta + \dot{\varphi}\right)^2 - Mgl\cos\theta \tag{68}$$

Since the variables ϕ and φ are cyclic, the conjugate momenta p_ϕ and p_φ are constant in time. Therefore,

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_1 \sin^2\theta + I_3 \cos^2\theta)\,\dot{\phi} + \mathrm{I}_3\ \dot{\varphi}\ \cos\theta = \alpha_2, \tag{69}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = I_3\,(\dot{\varphi} + \dot{\phi}\ \cos\theta) = \alpha_3, \tag{70}$$

and $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I_1\dot{\theta} \tag{71}$$

Since the system is conservative, the total energy E is constant in time. That is (see equ. 65 - Chapter 4), the Hamiltonian H is given by

$$H = E = T + V = \frac{p_\theta^2}{2I_1} + \frac{p_\varphi^2}{2I_3} + \frac{(p_\phi - p_\varphi\cos\theta)^2}{2I_1\sin^2\theta} + \mathrm{M}gl\cos\theta \tag{72}$$

where we have made use of equ. (71)

The characteristic function W in the present case can be written as (see equ. 29)

$$W = W_1(\theta) + \alpha_2\,\phi + \alpha_3\,\varphi \tag{73}$$

where

$$\alpha_2 = \frac{\partial W}{\partial \phi} = p_\phi, \quad \alpha_3 = \frac{\partial W}{\partial \varphi} = p_\varphi \tag{74}$$

The Hamilton - Jacobi equation, therefore, reads

$$\frac{1}{2I_1}\left(\frac{\partial W_1}{\partial \theta}\right)^2 + \frac{\alpha_3^2}{2I_3} + \frac{(\alpha_2 - \alpha_3 \cos\theta)^2}{2I_1 \sin^2\theta} + \text{Mg}l \cos\theta = \text{E}$$

in which $W_1(\theta)$ can be taken as $\Theta(\theta)$

The above equation can be integrated to give

$$\Theta(\theta) = \int \frac{\sqrt{2I_1 \sin^2\theta(E - A - Mgl\cos\theta) - (\alpha_2 - \alpha_3\cos\theta)^2}}{\sin\theta}\, d\theta$$

where $$A = \frac{\alpha_3^2}{2I_3}$$

Therefore,

$$\text{W} = \alpha_2\phi + \alpha_3\varphi + \int \frac{\sqrt{2I_1 \sin^2\theta(E - A - Mgl\cos\theta) - (\alpha_2 - \alpha_3\cos\theta)^2}}{\sin\theta}\, d\theta$$

With the above form of the Hamilton's characteristic function, the transformation equ. (24) yields,

$$t + \beta_1 = \frac{\partial W}{\partial E} = \int I_1 \sin\theta d\theta / \sqrt{2I_1 \sin^2\theta(E - A - Mgl\cos\theta) - (\alpha_2 - \alpha_3\cos\theta)^2}$$

or $(d\theta/dt)^2 I_1^2 \sin^2\theta = 2I_1 \sin^2\theta\,(E - A - Mgl\cos\theta) - (\alpha_2 - \alpha_3\cos\theta)^2$

With the change of variable to $u = \cos\theta$ and on introducing two new constants.

$\alpha = 2\,(E - A)/I_1, \beta = 2\,Mgl/I_1, b = \alpha_2/I_1, a = \alpha_3/I_1,$ (75)

one gets

$$(du/dt)^2 = (\alpha - \beta u)\,(1 - u^2) - (b - au)^2 \tag{76}$$

It is possible to describe the qualitative features of the motion by means of the above equation. Designating the r.h.s. of equ. (76) by $f(u)$, one gets

$$(du/dt)^2 = (1 - u^2)\,(\alpha - \beta u) - (b - au)^2 = f(u) \tag{77}$$

For u large, the dominant term in $f(u)$ is βu^3. Since β (see equ. 75) is always greater than zero, $f(u)$ is positive for large positive u and negative for large negative u. At the points $u = \pm 1$, $f(u)$ becomes equal to $-(b \mp a)^2$ and is therefore always negative, except for the unusual case where $u = \pm 1$ is a root (giving the vertical position of the top). Hence atleast one of the roots must lie in the region $u = u_3 > 1$, a region that does not correspond to real angles or that does not make sense. Truly speaking the motion of the top can occur only when $(du/dt)^2$ is positive, some where in the interval between $u = -1$ and $u = +1$, that is, θ lying between 0 and $+\pi$. We must conclude, therefore that for any actual top $f(u)$ will have two roots, u_1 and u_2 between -1 and $+1$ (see Fig. 8.1), and the top moves such that $\cos\theta$ always remain between these two roots that is, the inclination angles of the top with respect to the vertical line are limited between $\theta_1 = \cos^{-1}u_1$ and $\theta_2 = \cos^{-1} u_2$.

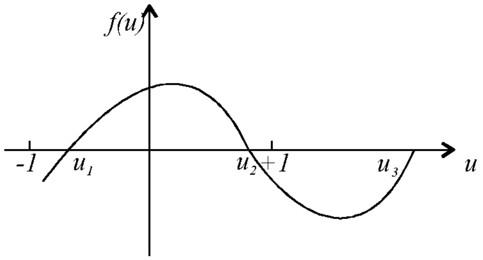

Fig. 8.1 Illustrating location of the turning angles of θ in the motion of a heavy symmetrical top.

8.2 The Hamiltonian in the present case is given by

$$H = T + V = p^2/2m + mg\,y = E, \tag{78}$$

the total energy of the system as H does not depend explicitly on time. Here y is taken in the vertical direction

The Hamilton – Jacobi equation in the present case, therefore, reads

$$\frac{1}{2m}\frac{\partial W}{\partial y} + mgy = E \tag{79}$$

From which

$$W = \sqrt{2m}\int \sqrt{(E - mgy)}\,dy + \text{constant of integration} \tag{80}$$

From equ. (24),

$$t + \beta = \frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial E} = (m/2)^{1/2} \int dy/(E\text{-}mgy)^{1/2}$$
$$= (m/2)^{1/2} \times 2/mg\,(E\text{-}mgy)^{1/2}$$

which gives

$$y = -\tfrac{1}{2}\,g\,(\beta + t)^2 + E/mg \tag{81}$$

Let the initial conditions be such that at $t = 0$, $y = y_0$ and $p = 0$. Hence from equ. (78) we get

$$E = mg\,y_0$$

which when substituted in equ. (81) gives

$$y = -\tfrac{1}{2}\,g\,(t + \beta)^2 + y_0 \tag{82}$$

Further since at $t = 0$, $y = y_0$, from the above equation, we get $\beta = 0$. Therefore, the equation of the path of particle falling vertically under the action of gravity is given by

$$y = y_0 - \tfrac{1}{2}\,gt^2$$

8.3 Let the origin be at the point of projection, x-axis parallel to the earth and y-axis perpendicular to it. At time t, suppose the projectile is at height y and at a distance x from the origin. Therefore, the kinetic and potential energies of the projectile are given by

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right), \ V = mg\,y$$

and hence

$$L = T - V = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - mg\,y,$$

and $$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

The Hamiltonian is, therefore

$$H = \tfrac{1}{2}m(p_x^2 + p_y^2) + mg\,y = E,$$

the total energy as the Hamiltonian H is not an explicit function of time. The Hamilton-Jacobi equation, therefore, reads

$$\tfrac{1}{2} m \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] + mgy = E \tag{83}$$

Using the method of separation of variables, let us put

$$W = W_x(x) + W_y(y)$$

in equ. (83). Doing so we get

$$\tfrac{1}{2} m \left[\left(\frac{dW_x}{dx}\right)^2 + \left(\frac{dW_y}{dy}\right)^2\right] + mgy = E,$$

giving

$$\frac{dW_x}{dx} = \sqrt{2m(E - mgy) - (dW_y/dy)^2}$$

Now *l.h.s.* depends on x and the r.h.s. on y but both are equal. This is only possible when both of them are equal to the same separation constant α_2, say

$$\frac{dW_x}{dx} = \sqrt{2m(E - mgy) - (dW_y/dy)^2} = \alpha_2$$

The Hamilton's characteristic function W then becomes

$$W = W_x + W_y = \alpha_2 x + \int \sqrt{2m(E - mgy) - \alpha_2^2}\, dy + \text{Const. of integration} \tag{84}$$

and as a result

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \frac{\partial W}{\partial E} = \int \frac{m\,dy}{\sqrt{2m(E - mgy) - \alpha_2^2}} \tag{85}$$

$$\text{and } \beta_2 = \frac{\partial W}{\partial \alpha_2} = x - \int \frac{\alpha_2\, dy}{\sqrt{2m(E - mgy) - \alpha_2^2}} \tag{86}$$

Solving the above equations simultaneously, we have for x

$$\beta_2 = x - \alpha_2/m\,(t + \beta_1)$$

or $$x = \alpha_2/m\,(t + \beta_1) + \beta_2 \tag{87}$$

From equ. (85), after integration, we get

$$t + \beta_1 = -1/mg\sqrt{2m(E - mgy) - \alpha_2^2} \tag{88}$$

From the conditions at $t = 0$,

$$E = ½\, m\ \upsilon_0^2,\ p_x = m\,v_0\cos\theta = m\,\dot{x}\ = \alpha_2 \text{ (from equ. 87)},$$

$$p_y = m\,v_0\sin\theta$$

Putting these values in equ. (88), we get

$$t + \beta_1 = -1/g\sqrt{\upsilon_0^2\sin^2\theta - 2gy} \tag{89}$$

Further at $t = 0$, $y = 0$, therefore, from above

$$\beta_1 = -v_0\sin\theta/g$$

With this, equ. (89) gives

$$y = -½\, gt^2 + v_0\sin\theta\, t \tag{90}$$

Further substituting the value of β_1 and also that at $t = 0$, $x = 0$ in equ. (87), we get

$$\beta_2 = -\alpha_2\beta_1/m = -\,mv_0\cos\theta \times -\frac{\upsilon_0\sin\theta}{g} \times 1/m$$

$$= v_0^2\sin\theta\cos\theta/g$$

Putting the value of $t + \beta_1$ from equ. (89) into (87), we get

$$\beta_2 = x + \frac{\alpha_2}{mg}\sqrt{\upsilon_0^2\sin^2\theta - 2gy}$$

Further putting the values of α_2 and β_2 in the above equation, we get the equation of trajectory as

$$x - \frac{\upsilon_0^2\sin\theta\cos\theta}{g} = -\frac{m\upsilon_0\cos\theta}{mg}\sqrt{\upsilon_0^2\sin^2\theta - 2gy}$$

or $$\left(x - \frac{\upsilon_0^2\sin\theta\cos\theta}{g}\right)^2 = \frac{\upsilon_0^2\cos^2\theta}{g^2}(\upsilon_0^2\sin^2\theta - 2gy)$$

$$= -\frac{2\upsilon_0^2\cos^2\theta}{g}\left(y - \frac{\upsilon_0^2\sin^2\theta}{2g}\right)$$

On simplification, we get

$$y = x\tan\theta - ½\, g\frac{x^2}{\upsilon_0^2\cos^2\theta}$$

which is equation of a parabola. Thus the problem is solved and the result being what one would expect.

8.4 From equ. (66) of the Kepler's problem, energy E is given by

$$E = \frac{-2\pi^2 mk^2}{(J_r + J_\theta)^2}$$

In the present case,

$k = e^2, J_r = \oint p_r\, dr = n_1 h$ (Given). $J_\theta = \oint p_\theta\, d\theta = n_2 h$ (given) where $n_1 = 0, 1, 2,$ .... is called the radial quantum number and $n_2 = 1, 2, 3,$ .... the azimuthal quantum number

Therefore, $$E = E_n = -\frac{2\pi^2 me^4}{n^2 h^2}$$

where $n = n_1 + n_2 = 1, 2, 3,$ .... is called the principal quantum number.

8.5 Let q_1, q_2, and q_3 represent the displacements along the axes q_1, q_2 and q_3, respectively. The Hamiltonian for a three-dimensional harmonic oscillator is then given by

$$H = (p_1^2/2m + \tfrac{1}{2} k_1 q_1^2) + (p_2^2/2m + \tfrac{1}{2} k_2 q_2^2) + (p_3^2/2m + \tfrac{1}{2} \mathrm{k}_3 q_3^2),$$

where p_1, p_2 and p_3 represent the conjugate momenta and k_1, k_2, nand k_3 are the force constants. The Hamilton-Jacobi equation in the present case would therefore read

$$1/2m\left[\left(\frac{\partial W}{\partial q_1}\right)^2 + \left(\frac{\partial W}{\partial q_2}\right)^2 + \left(\frac{\partial W}{\partial q_3}\right)^2\right] + \tfrac{1}{2}(k_1 q_1^2 + k_2 q_2^2 + k_3 q_3^2) = \alpha = E \qquad (91)$$

Let us use the method of separation of variables and putting

$$W = W_1(q_1) + W_2(q_2) + W_3(q_3)$$

in equ. (91), we get

$$[\tfrac{1}{2}m\left(\frac{dW_1}{dq_1}\right)^2 + \tfrac{1}{2} k_1 q_1^2] + [\tfrac{1}{2}m\left(\frac{dW_2}{dq_2}\right)^2 + \tfrac{1}{2} k_2 q_2^2] + [\tfrac{1}{2}m\left(\frac{dW_3}{dq_3}\right)^2 + \tfrac{1}{2} k_3 q_3^2]$$
$$= \alpha = E$$

The first part of the above equation is function of q_1, the second function of q_2 and the third function of q_3 and the sum is equal to E. This is only possible when each one is equal to a separation constant and the sum is equal to E. That is,

$$\tfrac{1}{2}m\left(\frac{dW_1}{dq_1}\right)^2 + \tfrac{1}{2} k_1 q_1^2 = \alpha_1 \qquad (92a)$$

$$\tfrac{1}{2}m\left(\frac{dW_2}{dq_2}\right)^2 + \tfrac{1}{2} k_2 q_2^2 = \alpha_2 \qquad (92b)$$

$$\tfrac{1}{2}m\left(\frac{dW_3}{dq_3}\right)^2 + \tfrac{1}{2} k_3 q_3^2 = \alpha_3 \qquad (92c)$$

and

$$E = \alpha_1 + \alpha_2 + \alpha_3$$

Therefore,

$$J_1 = \oint p_1\, dq_1 = \oint \frac{dW_1}{dq_1} dq_1 = \int \sqrt{2m\alpha_1 - mk_1 q_1^2}\, dq_1$$

which finally (See equ. 59) leads to

$$J_1 = \frac{2\pi\alpha_1}{\omega_1} \text{ or } a_1 = \frac{1}{2\pi} J_1 \omega_1 = \frac{J_1}{2\pi}\sqrt{k_1 / m}$$

Similarly we get

$$\alpha_2 = \frac{J_2}{2\pi}\sqrt{k_2 / m}, \text{ and } \alpha_3 = \frac{J_3}{2\pi}\sqrt{k_3 / m}$$

Therefore,

$$E = \alpha = \alpha_1 + \alpha_2 + \alpha_3 = H = \frac{J_1}{2\pi}\sqrt{k_1 / m} + \frac{J_2}{2\pi}\sqrt{k_2 / m} + \frac{J_3}{2\pi}\sqrt{k_3 / m}$$

which results in

$$\nu_1 = \frac{\partial H}{\partial J_1} = \frac{1}{2\pi}\sqrt{k_1 / m}$$

$$\nu_2 = \frac{\partial H}{\partial J_2} = \frac{1}{2\pi}\sqrt{k_2 / m}$$

and

$$\nu_3 = \frac{\partial H}{\partial J_3} = \frac{1}{2\pi}\sqrt{k_3 / m}$$

For $k_1 = k_2 = k_3 = k, \nu_1 = \nu_2 = \nu_3$, the oscillator becomes isotropic and the motion is completely degenerate.

8.6 The kinetic and potential energies of a simple pendulum (see Fig. 6.2) consisting of mass point m attached at the end of a weightless rod (or string) of length l, are given by

$$T = \tfrac{1}{2} I\dot{q}^2,$$

and

$$V = mgl\,(1\text{-}\cos q)$$

where q is the displacement of the mass point m and I is the moment of inertia of the pendulums about an axis passing through the point of suspension O and perpendicular to the plane of the page.

Expanding the cos q term by Taylor series and in the small amplitude limit dropping all but the first term in the expansion, we get

$$V = mgl\left(1 - 1 + \frac{q^2}{2}\right) = \tfrac{1}{2}\, mglq^2$$

As a result, the Lagrangian

$$L = T - V = \tfrac{1}{2} I\dot{q}^2 - \tfrac{1}{2}\, mg\, lq^2$$

and hence the momentum conjugate to q,

$$p_q = \frac{\partial L}{\partial \dot{q}} = I\dot{q}$$

Therefore, the Hamiltonian

$$H = T + V = p_q^2/2I + ½\, mgl\, q^2 = \alpha = \mathrm{E}$$

as H is not an explicit function of time. The Hamilton - Jacobi equation is, then given by

$$1/2I\left(\frac{\partial W}{\partial q}\right)^2 + ½\, mgl\, q^2 = E$$

from which

$$\frac{dW}{dq} = \sqrt{2I(E - ½mglq^2)}$$

As a result, the action variable

$$J = \oint p_q\, dq = \oint \frac{dW}{dq}\, dq = \oint \sqrt{\mathrm{Im}gl}\sqrt{2E/mgl - q^2}\, dq$$

The substitution

$$q = \sqrt{2E/mgl}\, \sin\theta,$$

results in

$$J = \sqrt{\mathrm{Im}gl}\ (2E/mgl)\oint \cos^2\theta d\theta$$

$$= \sqrt{I/mgl}\ 2E\int_0^{2\pi} \cos^2\theta d\theta$$

$$= \sqrt{I/mgl}\ 2E\,\pi$$

Therefore,

$$\alpha = E = H = J/2\pi\ \sqrt{mgl/I}\ = J/2\pi\sqrt{mgl/ml^2}$$

$$= J/2\pi\ \sqrt{g/l}$$

The frequency of oscillations,

$$\nu = \frac{\partial H}{\partial J} = ½\pi\sqrt{g/l}$$

and hence $E = J\nu$

8.7 We would now examine the motion in space, rather than making use of our knowledge that the orbit lies in plane. In terms of the spherical polar coordinates (r, θ, ϕ) the Kepler problem becomes a special case of the general treatment for central force motion in space. The potential and kinetic energies in the present case are given by

$$V(r) = -k/r,$$

and

$$T = ½\, m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

we shall confine our discussion to the bound case, that is $E < 0$. Hence the motion in each of the coordinates would be periodic, liberation in r and θ, and rotation in ϕ. The Lagrangian.

$$L = T - V = \tfrac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2\right) + k/r$$

and therefore, the momenta conjugate to r, θ, and ϕ would be given by

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, p_\phi = mr^2 \sin^2\theta\,\dot{\phi}$$

The Hamiltonian as a result becomes

$$H = T + V = 1/2\,m \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - k/r \tag{93}$$

and hence the Hamilton-Jacobi equation reads

$$1/2\,m\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2\right] - k/r = \alpha = E, \tag{94}$$

the total energy as the Hamiltonian is not an explicit function of time.

Further using the method of separation of variables and substituting

$$W = W_r(r) + W_\theta(\theta) + W_\phi(\phi)$$

in equ. (94), we get

$$\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial W\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta}\right] - 2\,mk/r = 2mE$$

where we have put

$$p_\phi = \frac{\partial W\phi}{\partial \phi} = \alpha_\phi, \tag{95}$$

a constant of motion (angular momentum conjugate to ϕ or about the polar axis) as ϕ is cyclic in Hamiltonian. Thus

$$\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2, \text{ a constant} \tag{96}$$

and the dependence of W on r is given by the remains of the Hamilton-Jacobi equation,

$$\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} = 2\,m\,(E + k/r) \tag{97}$$

To identify α_θ, equ. (96) can be writens as

$$p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} = \alpha_\theta^2,$$

so that the Hamiltonian, equ. (93) appears as

$$H = 1/2\,m\left(p_r^2 + \frac{\alpha_\theta^2}{r^2}\right) - k/r$$

Comparison with equ. (42) for the Hamiltonian as expressed in terms of the plane polar coordinates in the plane of the orbit shows that α_θ is same as p_θ, the magnitude of the total angular momentum, that is,

$$\alpha_\theta = p_\theta = l$$

From equs. (95) - (97), the action variables are given by

$$J_\phi = \oint \frac{\partial W}{\partial \phi}\, d\phi = \oint \alpha_\phi d\phi, \tag{98a}$$

$$J_\theta = \oint \frac{\partial W}{\partial \theta}\, d\theta = \oint \sqrt{\alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2\theta}}\, d\theta, \tag{98b}$$

and

$$J_r = \oint \frac{\partial W}{\partial r}\, dr = \oint \sqrt{2mE + \frac{2mk}{r} - \frac{\alpha_\theta^2}{r^2}}\, dr \tag{98c}$$

The first integral is simple, ϕ goes through 2π radians in a complete revolution, therefore,

$$\mathrm{J}_\phi = 2\pi\, \alpha_\phi \tag{99}$$

The 2nd integral (98b) gives

$$J_\theta = 2\pi\, (\alpha_\theta - \alpha_\phi), \tag{100}$$

from which and from equ. (99),

$$\alpha_\theta = (J_\theta + J_\phi)/2\pi$$

The last integral (98c) can, therefore, be written as

$$J_r = \oint \sqrt{2mE + \frac{2mk}{r} - \frac{(J_\theta + J_\phi)^2}{4\pi^2 r^2}}\, dr$$

which we would write as

$$J_r = \oint \sqrt{-A + 2B/r - C/r^2}\, dr,$$

where

$$A = -2mE,\ B = mk, \text{ and } C = (J_\theta + J_\phi)^2/4\,\pi^2$$

The total integral results in

$$J_r = 2\pi\left(B \quad \sqrt{A} \quad \sqrt{C}\right)$$

and upon substituting the coefficients *A, B* and *C*, we get

$$J_r = 2\pi\left[\frac{mk}{\sqrt{-2mE}} - \frac{(J_\theta + J_\phi)}{2\pi}\right]$$

$$= -(J_\theta + J_\phi) + \pi k\sqrt{2m/-E}$$

Solving for *E*, we have

$$H = E = -\frac{2\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^2} \tag{101}$$

Note that all three of the action variables appear only in the form $J_r + J_\theta + J_\phi$. Therefore, all the three frequencies

$$\nu = \frac{\partial H}{\partial J_r} = \frac{\partial H}{\partial J_\theta} = \frac{\partial H}{\partial J_\phi} = \frac{4\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^3} \tag{102}$$

are equal or in other words the motion is degenerate.

If we evaluate the sum of J's, that is, $J_r + J_\theta + J_\phi$ in terms of the energy E from equ. (101), the period of orbit is

$$T = 1/\nu = \pi k\sqrt{m/-2E^3} \tag{103}$$

If we use the result, $a = -k/2E$, we get

$$T^2 = (4\pi^2 m/k)a^3$$

which is nothing but Kepler's third law

The three quantum conditions being

$$J_r = n_r h,\ J_\theta = n_\theta h, \text{ and } J_\phi = n_\phi h,$$

the quantized energy levels from equ. (101) are given by

$$E = E_n = -\frac{2\pi^2 mk^2}{n^2h^2} = -\frac{2\pi^2 me^4}{n^2h^2}$$

where we have put $n = n_r + n_\theta + n_\phi$ and $k = e^2$ for the case of H-atom.

8.8 The Hamiltonian in the present case is given by

$$H = \sqrt{c^2p^2 + m_0^2c^4} - k/r$$

Since in the plane polar coordinates, $p^2 = p_r^2 + p_\theta^2/r^2$, therefore,

$$H = \sqrt{c^2(p_r^2 + 1/r^2 p_\theta^2) + m_0^2c^4} - k/r$$

and hence the Hamilton - Jacobi equation reads

$$\sqrt{c^2\left\{\left(\frac{\partial W}{\partial r}\right)^2 + 1/r^2\left(\frac{\partial W}{\partial \theta}\right)^2\right\} + m_0^2c^4} - k/r = \alpha = E \tag{104}$$

as the Hamiltonian is not an explicit function of time.

Further using the method of separation of variables and substituting

$$W = W_r(r) + W_\theta(\theta)$$

in equ. (104), we get

$$\sqrt{c^2\left\{\left(\frac{\partial W_r}{\partial r}\right)^2 + 1/r^2\left(\frac{\partial W_\theta}{\partial \theta}\right)^2\right\} + m_0^2c^4} - k/r = E$$

Since

$$\frac{\partial W_\theta}{\partial \theta} = \alpha_\theta, \text{ a constant} \tag{105}$$

hence

$$\frac{\partial W_r}{\partial r} = 1/c\sqrt{(E+k/r)^2 - m_0^2c^4 - \frac{c^2\alpha_\theta^2}{r^2}} \tag{106}$$

The action variables as a result are given by

$$J_\theta = \oint p_\theta d\theta = \oint \frac{\partial W_\theta}{\partial \theta} d\theta = 2\pi\ \alpha_\theta \text{ or } \alpha_\theta = J_\theta/2\pi \tag{107}$$

and

$$J_r = \oint p_r dr = \oint \frac{\partial W_r}{\partial r} dr$$

$$= 1/c\oint\sqrt{-(m_0^2c^4 - E^2) + \frac{2kE}{r} - \frac{(c^2\alpha_\theta^2 - k^2)}{r^2}}$$

which we would write as

$$J_r = 1/c\oint\sqrt{-A + \frac{2B}{r} - \frac{D}{r^2}}\, dr,$$

where

$$A = m_0^2c^4 - E^2, B = kE, D = c^2\alpha_\theta^2 - k^2$$

The total integral results in

$$J_r = 1/c\ 2\pi\left(\frac{B}{\sqrt{A}} - \sqrt{D}\right)$$

and upon substituting the coefficients A, B and D, we get

$$J_r = 1/c\ 2\pi\left[\frac{kE}{\sqrt{m_0^2c^4 - E^2}} - \sqrt{c^2\alpha_\theta^2 - k^2}\right]$$

or

$$J_r + 2\pi/\mathrm{c}\ \sqrt{c^2J_\theta^2/4\pi^2 - k^2} = \frac{2\pi kE/c}{\sqrt{m_0^2c^4 - E^2}}$$

or

$$\left[J_r + \sqrt{J_\theta^2 - \frac{4\pi^2k^2}{c^2}}\right]/2\pi k/c = E/\sqrt{m_0^2c^4 - E^2}$$

Squaring both the sides, we get

$$(4\pi^2/k^2/c^2)\ /\left(J_r + \sqrt{J_\theta^2 - \frac{4\pi^2k^2}{c^2}}\right)^2 = \frac{m_0^2c^4 - E^2}{E^2} = \frac{m_0^2c^4}{E^2} - 1$$

and hence

$$E/m_0c^2 = 1/\left[1 + \frac{4\pi^2k^2}{(cJ_r + \sqrt{c^2J_\theta^2 - 4\pi^2k^2})^2}\right]^{1/2} \tag{108}$$

9

CHAPTER

Lagrangian and Hamiltonian Formulation for Continuous Systems

9.1 INTRODUCTION

The formulations of mechanics which we have discussed so far were devised for treating systems having a finite number of degrees of freedom. There are, however, some mechanical problems involving continuous systems which are characterized by an infinite number of degrees of freedom. A typical example of such a system being the problem of a vibrating elastic solid. Here each point of the continuous solid takes part in the oscillations and the complete motion of the system can be described by specifying the position coordinates of all the points. We would now try to modify the previous formulations of mechanics in order to handle such problems. The most convenient method to do so is to approximate the continuous system by the one containing discrete particles and then examine the change in the equation of motion as the continuous limit is approached.

9.2 THE TRANSITION FROM A DISCRETE TO A CONTINUOUS SYSTEM

Consider an infinitely long elastic rod which can undergo small longitudinal vibrations (or oscillatory displacements) of the particles of the rod. Such a system can be approximated by an infinite chain of equal mass points spaced equally at a distance a apart and connected by uniform massless springs having force constants k (see Fig. 9.1). It would be assumed that the mass points can move only along the length of the chain or along the axis of the rod. Denoting the displacement of the ith particle from its equilibrium position by φ_i, the kinetic energy is given by

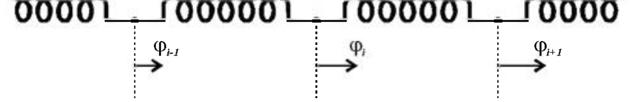

Fig. 9.1 A discrete system of equal mass points connected by springs.

$$T = \tfrac{1}{2}\sum_i m\dot{\varphi}_i^2 \tag{1}$$

where m is the mass of each particle. The corresponding potential energy is the sum of the potential energies of each spring as a result of displacement from ith equilibrium position and is given by

$$V = \tfrac{1}{2}\sum_i k(\varphi_{i+1} - \varphi_i)^2 \tag{2}$$

As a result, the Lagrangian of the system is

$$\begin{aligned} L = T - V &= \tfrac{1}{2}\sum_i [m\dot{\varphi}_i^2 - k(\varphi_{i+1} - \varphi_i)^2] \\ &= \tfrac{1}{2}\sum_i a\left[m/a\,\dot{\varphi}_i^2 - ka\left(\frac{\varphi_{i+1} - \varphi_i}{a}\right)^2\right] \\ &= \sum_i a\,L_i \end{aligned} \tag{3}$$

The resulting Lagrange's equations of motion for the coordinates φ_i are

$$m/a\,\ddot{\varphi}_i - ka\frac{\varphi_{i+1} - \varphi_i}{a^2} + ka\frac{\varphi_i - \varphi_{i-1}}{a^2} = 0 \tag{4}$$

The above form of Lagrangian in equ. (3) and corresponding form of the equations of motion (4) has been chosen for convenience so that when $a \to 0$, we can go over to a continuous rod. It is clear that m/a reduces to μ, the mass per unit length of the continuous rod but the limiting value of ka is not so obvious. Let us try to interpret ka. For an elastic rod obeying Hooke's law, we know that the extension of the rod per unit length is proportional to the force exerted on the rod, that is,

$$F = Y\xi \tag{5}$$

where ξ is the elongation per unit length and Y the Young's modulus. Now the extension of a length a of a discrete system per unit length will be $\xi = (\varphi_{i+1} - \varphi_i)/a$. The force necessary to stretch the spring by this amount is

$$F = k(\varphi_{i+1} - \varphi_i) = ka\left(\frac{\varphi_{i+1} - \varphi_i}{a}\right) \tag{5$'$}$$

Comparison of equs. (5) and (5′) shows that ka must correspond to the Young's modulus of the continuous rod. In going from the discrete to the continuous case, the integer index i identifying the particular mass point becomes the continuous coordinate x, instead of the variable φ_i we have $\varphi(x)$. Further the quantity

$$\frac{\varphi_{i+1} - \varphi_i}{a} = \frac{\varphi(x + a) - \varphi(x)}{a}$$

occuring in L_i obviously approaches the limit $d\varphi/dx$ as 'a', playing the role of dx, approaches zero. Finally, the summation over a discrete number of particles becomes an integral over x, the length of the rod. As a result, the Lagrangian (3) in continuous limit becomes

$$L = \tfrac{1}{2}\int\left[\mu\dot{\varphi}^2 - Y\left(\frac{d\varphi}{dx}\right)^2\right]dx \tag{6}$$

In the limit as $a \to 0$, the last two terms in the equation of motion (4) become

$$\lim_{a\to 0} -Y/a\left[\left(\frac{d\varphi}{dx}\right)_x - \left(\frac{d\varphi}{dx}\right)_{x-a}\right]$$

which is nothing but $-Y\times$ second derivative of φ. Therefore, the equation of motion for the continuous elastic rod is

$$\mu d^2\varphi/dt^2 - Y\, d^2\varphi/dx^2 = 0 \tag{7}$$

the usual wave equation with the propagation velocity (of the longitudinal elastic waves)

$$v = \sqrt{Y/\mu} \tag{8}$$

The above example supports the procedure used here and at the same time provides a clue to a procedure for the treatment of problems involving continuous media. Note that here the continuous variable x, the position coordinate has replaced the discrete index i. Just as each value of i corresponded to a different φ_i of the system, therefore, here for each x, there is a different $\varphi(x)$. Since φ depends also on the continuous variable t, we should write $\varphi(x, t)$ instead of $\varphi(x)$.

Equ. (6) also shows that the Lagrangian appears as an integral over the continuous variable x and therefore it would have the form

$$L = \int \pounds dx \tag{9}$$

where £ is known as the Lagrangian density. For the longitudinal vibrations of the continuous rod the Lagrangian density is

$$\pounds = \tfrac{1}{2}\left\{\mu\dot{\varphi}^2 - Y\left(\frac{d\varphi}{dx}\right)^2\right\} \tag{10}$$

corresponding to the continuous limit of the quantity L_i appearing in equ. (3). The extension to the three dimensional case is obvious and the Lagrangian would have the form

$$L = \int\int\int \pounds\, dx\, dy\, dz \tag{11}$$

It is, truly speaking the Lagrangian density rather than the Lagrangian itself, which would be used to describe the motion of the system.

9.3 THE LAGRANGIAN FORMULATION FOR CONTINUOUS SYSTEMS

It may be noted from equ. (10) that the Lagrangian density £ for the elastic rod is a function of $\dot{\varphi} = d\varphi/dt$ and as well as of $d\varphi/dx$. In the general case £ might well be an explicit function of x, t and also of φ itself. Therefore, the Lagrangian density for any one-dimensional continuous system would appear as a function of the form

$$\pounds = \pounds\left(\varphi, \frac{d\varphi}{dx}, \frac{d\varphi}{dt}, x, t\right) \tag{12}$$

The Lagrangian *L*, following equ. (9), is then the integral of £ over the range of *x* defining the system, and Hamilton's principle in the limit of the continuous system appears as

$$\delta I = \delta \int_1^2 \int \pounds \, dx\, dt = 0 \tag{13}$$

It may be noted here that in case the Hamilton's principle for a continuous system is to have any usefulness, it must be possible to derive equ. (7) directly by variation of the double integral of £ in equ. (13). We can carry out this variation by methods differing only slightly from those used earlier for a discrete system. The variation is only on φ and its derivatives $\partial\varphi/\partial t$ and $\partial\varphi/\partial x$, the parameters *x* and t are not affected by the variation either directly or in the ranges of integration. Just as the variation of φ is taken to be zero at the end points t_1 and t_2, so the variation of φ at the limits x_1 and x_2 of the integration in *x* is also to be zero.

In analogous to the relation (13), Hamilton's principle in the three-dimensional case would have the form

$$\delta I = \delta \int_1^2 \iiint \pounds \, dx_1\, dx_2\, dx_3\, dt = 0 \tag{14}$$

in which the Lagrangian density appears as a functions of the form

$$\pounds = \pounds\left(\varphi, \dot{\varphi}, \frac{d\varphi}{dx_i}\right), i = 1, 2, 3 \tag{15}$$

The variation in the Lagrangian density £ can be written as

$$\delta \pounds = \frac{\partial \pounds}{\partial \varphi}\delta\varphi + \frac{\partial \pounds}{\partial \dot{\varphi}}\delta\dot{\varphi} + \sum_{i=1}^{3} \frac{\partial \pounds}{\partial(\frac{\partial \varphi}{\partial x_i})}\delta\left(\frac{\partial \varphi}{\partial x_i}\right) \tag{16}$$

Hamilton's principle as a result, takes the form

$$\int_1^2 \iiint \left[\frac{\partial \pounds}{\partial \varphi}\delta\varphi + \frac{\partial \pounds}{\partial \dot{\varphi}}\delta\dot{\varphi} + \sum_{i=1}^{3} \frac{\partial \pounds}{\partial(\frac{\partial \varphi}{\partial x_i})}\delta\left(\frac{\partial \varphi}{\partial x_i}\right)\right] dx_1 dx_2 dx_3 dt = 0 \tag{17}$$

Interchanging δ and $\partial/\partial t$ as they are independent of each other and then carrying out integration by parts, the second term in equ. (17) becomes

$$\int_1^2 \frac{\partial \pounds}{\partial \dot{\varphi}}\delta\dot{\varphi}\, dt = \int_1^2 \frac{\partial \pounds}{\partial \dot{\varphi}}\, \partial/\partial t(\delta\varphi)\, dt = \frac{\partial \pounds}{\partial \dot{\varphi}}\delta\varphi \Big|_{t_1}^{t_2} - \int_1^2 \frac{d}{dt}\left(\frac{\partial \pounds}{\partial \dot{\varphi}}\right)\delta\varphi\, dt$$

$$= \int_1^2 \frac{d}{dt}\left(\frac{\partial \pounds}{\partial \dot{\varphi}}\right)\delta\varphi\, dt$$

as the first term on the r.h.s. of the above equation vanishes because of the fact that $\delta\varphi$ vanishes at the end points. Further interchanging δ and $\partial/\partial x_i$ as they are independent of each other and then carrying out the integration by parts, the third term in equ. (17) becomes

$$\int \frac{\partial £}{\partial\left(\frac{\partial \varphi}{\partial x_i}\right)} \delta\left(\frac{\partial \varphi}{\partial x_i}\right) dx_i = \int \frac{\partial £}{\partial\left(\frac{\partial \varphi}{\partial x_i}\right)} \partial/\partial x_i\,(\delta\varphi) dx_i$$

$$= \frac{\partial £}{\partial\left(\frac{\partial \varphi}{\partial x_i}\right)} \delta\varphi - \int \frac{d}{dx_i}\left(\frac{\partial £}{\partial\left(\frac{\partial \varphi}{\partial x_i}\right)}\right) \delta\varphi\, dx_i$$

$$= -\int \frac{d}{dx_i}\left(\frac{\partial £}{\partial\left(\frac{\partial \varphi}{\partial x_i}\right)}\right) \delta\varphi\, dx_i$$

as the first term on the r.h.s. of the above equation vanishes because of the fact that $\delta\varphi$ vanishes at the end values of x_i.

Incorporating the above calculations; equ. (17) takes on the form

$$\int_1^2 \iiint \delta\varphi \left[\frac{\partial £}{\partial \varphi} - \frac{d}{dt}\left(\frac{\partial £}{\partial \dot{\varphi}}\right) - \sum_{i=1}^{3} \frac{d}{dx_i}\left(\frac{\partial £}{\partial(\frac{\partial \varphi}{\partial x_i})}\right)\right] dx_1\, dx_2\, dx_3\, dt = 0$$

The arbitrary nature of $\delta\varphi$ (except at the boundaries) implies that the integral can vanish only when the integrand itself is zero, yielding the equation of motion.

$$\frac{d}{dt} \frac{\partial £}{\partial(\frac{\partial \varphi}{\partial t})} + \sum_{i=1}^{3} \frac{d}{dx_i}\left(\frac{\partial £}{\partial(\frac{\partial \varphi}{\partial x_i})}\right) - \frac{\partial £}{\partial \varphi} = 0 \tag{18}$$

which is a partial differential equation in four continuous variables x_1, x_2, x_3 and t, yielding φ as $\varphi(x_1, x_2, x_3, t)$

For the specific case of the longitudinal vibrations in an elastic rod, it is seen from the form of the Lagrangian density (see equ. 10).

$$£ = ½\left\{\mu\left(\frac{\partial \varphi}{\partial t}\right)^2 - Y\left(\frac{\partial £}{\partial x}\right)^2\right\}$$

that $\dfrac{\partial £}{\partial \varphi} = 0, \dfrac{\partial £}{\partial\left(\frac{\partial \varphi}{\partial t}\right)} = \mu \dfrac{\partial \varphi}{\partial t}, \dfrac{\partial £}{\partial\left(\frac{\partial \varphi}{\partial x}\right)} = -Y \dfrac{\partial \varphi}{\partial x}$

Substituting above into equ. (18) leads to

$$\mu\frac{d^2\varphi}{dt^2} - Y\frac{d^2\varphi}{dx^2} = 0,$$

the usual wave equation or the equation of motion for the continuous elastic rod (see equ. 7).

It may be noted that the form of the Lagrangian density and the subsequent equation of motion, have been obtained assuming that each point in the system can suffer only one type of displacement indicated by φ. In a more complicated problems, such as the vibrations of an elastic solid, a particle would, of course, undergo displacements along all the three directions. In such a case, there would be three types of generalized coordinates which may be denoted by an integer index j, that is, $\varphi_j(x_i, t), j = 1, 2, 3$. In such a case, for each $\varphi_j(x_i, t)$, there would be an equation of motion of the form

$$\frac{d}{dt}\frac{\partial £}{\partial \dot{\varphi}_j} - \sum_i \frac{d}{dx_i}\left(\frac{\partial £}{\partial\left(\frac{\partial \varphi_j}{\partial x_i}\right)}\right) - \frac{\partial £}{\partial \varphi_j} = 0, \mathrm{j} = 1, 2, 3 \tag{19}$$

It may further be noted that while deriving equ. (18), we have treated x_i and t at equal footing. This equal status of space and time reminds us of the special relativity. In fact the entire procedure slips very easily into a Lorentz covariant formulation. The product $dx_1\, dx_2\, dx_3\, dt$ is essentially an element of volume in world space and hence is invariant under Lorentz transformation. The equation of motion (18), in the covariant form becomes simply

$$\sum_{\mu=1}^{4} \frac{d}{dx_\mu}\left(\frac{\partial £}{\partial\left(\frac{\partial \varphi}{\partial x_\mu}\right)}\right) - \frac{\partial £}{\partial \varphi} = 0 \tag{20}$$

which would be relativistically invariant (or invariant w.r.t. Lorentz transformation) provided £ and φ are world scalars.

In the general case of a elastic solid, the covariant form of the equation of motion (19), would appear as

$$\sum_{\mu=1}^{4} \frac{d}{dx_\mu}\left(\frac{\partial £}{\partial\left(\frac{\partial \varphi_j}{\partial x_\mu}\right)}\right) - \frac{\partial £}{\partial \varphi_j} = 0, \mathrm{j} = 1, 2, 3 \tag{21}$$

9.4 HAMILTON'S EQUATIONS OF MOTION FOR CONTINUOUS SYSTEMS

In analogous to the Lagrangian formulation for continuous systems, we would express the Hamiltonian of the system (continuous rod) as

$$H = \int\int\int K\, dx_1\, dx_2\, dx_3, \tag{22}$$

where K is called the Hamiltonian density and the integration is taken over the entire space.

In the continuous limit (or limit of the continuous rod), the momentum density in analogous to the canonical momenta $p_i = \dfrac{\partial L}{\partial \dot{\varphi}_i}$ in the case of discrete system (a linear chain of mass points) can be defined as

$$\pi = \frac{\partial \pounds}{\partial \dot{\varphi}}$$

and similarly the Hamiltonian density analogous to the Hamiltonian $H = \sum_i p_i \dot{q}_i - L$ in the discrete system can be defined as

$$K = \pi\dot{\varphi} - \pounds$$

or

$$\pounds = \pi\dot{\varphi} - K \tag{24}$$

Hamilton's principle in the limit of the continuous rod takes the form

$$\delta \int_1^2 \int\int\int \pounds\, dx_1\, dx_2\, dx_3\, dt = 0 \tag{25a}$$

where of course $\delta\varphi$ and $\delta\pi$ vanish at the end points, and at the end times, that is,

$$\delta\varphi(x_i, t_1) = \delta\varphi(x_i, t_2) = 0 \tag{25b}$$

and

$$\delta\pi(x_i, t_1) = \delta\pi(x_i, t_2) = 0 \tag{25c}$$

Putting the value of the Lagrangian density from equ. (24) into equ. (25*a*), we get

$$\int_1^2 \int\int\int \delta(\pi\dot{\varphi} - K) dx_1\, dx_2\, dx_3\, dt = 0$$

or

$$\int_1^2 \int\int\int (\pi\delta\dot{\varphi} + \dot{\varphi}\,\delta\pi - \delta K) dx_1\, dx_2\, dx_3\, dt = 0 \tag{26}$$

in which the Hamiltonian density appears as a function of the form

$$K = K\left(\varphi, \pi, \frac{\partial\varphi}{\partial x_i}, \frac{\partial\pi}{\partial x_i}\right), \quad i = 1, 2, 3 \tag{27}$$

whose variation can be written as

$$\delta K = \frac{\partial K}{\partial\varphi}\delta\varphi + \frac{\partial K}{\partial\pi}\delta\pi + \sum_{i=1}^{3} \frac{\partial K}{\partial\left(\dfrac{\partial\varphi}{\partial x_i}\right)}\delta\left(\frac{\partial\varphi}{\partial x_i}\right) + \sum_{i=1}^{3} \frac{\partial K}{\partial\left(\dfrac{\partial\pi}{\partial x_i}\right)}\delta\left(\frac{\partial\pi}{\partial x_i}\right)$$

Hamilton's principle, therefore, takes the form

$$\int_1^2 \iiint \left[\pi\delta\dot{\varphi} + \dot{\varphi}\delta\pi - \frac{\partial K}{\partial\varphi}\delta\varphi - \frac{\partial K}{\partial\pi}\delta\pi - \sum_{i=1}^{3} \frac{\partial K}{\partial\left(\dfrac{\partial\varphi}{\partial x_i}\right)} \delta\left(\frac{\partial\varphi}{\partial x_i}\right) \right.$$

$$\left. - \sum_{i=1}^{3} \frac{\partial K}{\partial\left(\dfrac{\partial\pi}{\partial x_i}\right)} \delta\left(\frac{\partial\pi}{\partial x_i}\right) \right] dx_1 dx_2 dx_3 dt = 0 \qquad (28)$$

Interchanging δ and $\partial/\partial t$ as they are independent of each other and then carrying out the integration by parts, the first term in the above equation becomes

$$\int_1^2 \pi\delta\dot{\varphi}dt = \int_1^2 \pi\,\partial/\partial t\,(\delta\varphi)dt = \pi\delta\varphi\,\Big|_1^2 - \int_1^2 \dot{\pi}\delta\varphi\, dt = -\int_1^2 \dot{\pi}\delta\varphi dt \qquad (29a)$$

as the first term on the r.h.s. vanishes because of the fact that $\delta\varphi$ vanishes at t_1 and t_2. Further interchanging δ and $\partial/\partial x_i$ as they are independent of each other and then carrying out the integration by parts, the fifth term in equ. (28) becomes

$$\int \frac{\partial K}{\partial\left(\dfrac{\partial\varphi}{\partial x_i}\right)} \frac{\partial}{\partial x_i}(\delta\varphi)dx_i = \frac{\partial K}{\partial\left(\dfrac{\partial\varphi}{\partial x_i}\right)}\delta\varphi - \int \frac{d}{dx_i}\left(\frac{\partial K}{\partial\left(\dfrac{\partial\varphi}{\partial x_i}\right)}\right)\delta\varphi dx_i$$

$$= -\int \frac{d}{dx_i}\left(\frac{\partial K}{\partial\left(\dfrac{\partial\varphi}{\partial x_i}\right)}\right)\delta\varphi dx_i \qquad (29b)$$

as the first term on the r.h.s. of the above equation vanishes because of the fact that $\delta\varphi$ vanishes at the ends of x_i. In the similar fashion, the 6th term in equ. (28) becomes

$$\int \frac{\partial K}{\partial\left(\dfrac{\partial\pi}{\partial x_i}\right)} \delta\left(\frac{\partial\pi}{\partial x_i}\right) dx_i = -\int \frac{d}{dx_i}\left(\frac{\partial K}{\partial\left(\dfrac{\partial\pi}{\partial x_i}\right)}\right)\delta\pi\, dx_i \qquad (29c)$$

Substituting (29 *a*, *b*, *c*) into equ. (28) and then collecting the coefficients of $\delta\varphi$ and $\delta\pi$, we get

$$\int_1^2 \iiint \left[\delta\varphi\left(-\dot{\pi} - \frac{\partial K}{\partial\varphi} + \nabla\cdot\frac{\partial K}{\partial\nabla\varphi}\right) + \delta\pi\left(\dot{\varphi} - \frac{\partial K}{\partial\pi} + \nabla\cdot\frac{\partial K}{\partial\nabla\pi}\right)\right] dt\, dx_1 dx_2 dx_3 = 0$$

Now $\delta\varphi$ and $\delta\pi$ being arbitrary except at the ends, the above integral would be identically zero only if the coefficients of $\delta\varphi$ and $\delta\pi$ are separately zero, that is,

$$\dot{\varphi} = \frac{\partial K}{\partial\pi} - \nabla\cdot\frac{\partial K}{\partial(\nabla\pi)} = \frac{\not{\partial} K}{\partial\pi} \qquad (30)$$

and $$\dot{\pi} = -\left(\frac{\partial K}{\partial \varphi} - \nabla \cdot \frac{\partial K}{\partial \nabla \varphi}\right) = -\frac{\delta K}{\partial \pi}$$

These are Hamilton's equations of motion for continuous systems. We have here used the notion of functional derivatives $\frac{\delta K}{\partial \pi}$ and $\frac{\delta K}{\partial \varphi}$ so that the equations of motion resemble with the equations of motion for discrete systems.

9.5 LAGRANGE'S EQUATION OF MOTION FOR THE TRANSVERSE VIBRATIONS OF THE STRING

Consider a series of mass points (or atoms) connected so as to form a string under tension T (see Fig. 9.2). Let φ_{i-1}, φ_i, φ_{i+1} represent the transverse displacements of the (i–1)th, ith and (i+1)th mass points, respectively. For the particular atom i, the net transverse force acting on it is

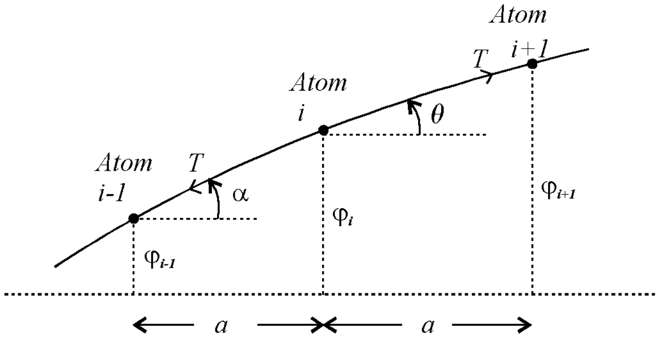

Fig. 9.2 Transverse displacements for line of connected mass points under tension

$$T\sin\theta - T\sin\alpha = T(\sin\theta - \sin\alpha) = T(\theta - \alpha)$$

for small displacements. Further nothing that

$$\sin\theta \approx \tan\theta \approx \theta = \frac{\varphi_{i+1} - \varphi_i}{a}$$

and $$\sin\alpha \approx \tan\alpha \approx \alpha = \frac{\varphi_i - \varphi_{i-1}}{a}$$

where a is the separation between the two consecutive atoms. As a result, the equation of motion for the ith atom is

$$m\ddot{\varphi}_i = T(\theta - \alpha) = T/a\,(\varphi_{i+1} + \varphi_{i-1} - 2\varphi_i) \tag{31}$$

from Newton's second law. This is identical with equ. (4) derived earlier in the case of longitudinal vibrations if we identify the spring constant k with T/a. One, of course, notices that φ_i here corresponds to transverse displacement and not to the previous longitudinal one.

The Lagrangian for the line of mass points (or atoms) can, therefore, be written as

$$L = ½\ \sum_i \left[m\dot{\varphi}_i^2 - k\left(\varphi_{i+1} - \varphi_i\right)^2 \right]$$

$$= ½\ a \sum_i \left[m/a\ \dot{\varphi}_i^2 - ka\left(\frac{\varphi_{i+1} - \varphi_i}{a} \right)^2 \right] \qquad (32)$$

In the continuous limit, the above expression takes the form

$$L = ½ \int \left[\mu\dot{\varphi}^2 - T\left(\frac{\partial\varphi}{\partial x} \right)^2 \right] dx \qquad (33)$$

where we have utilized $k = T/a$ (or $ka = T$), $\mu = m/a$ and replaced summation by integration and a by dx. Further, in one dimension, we can define Lagrangian density £ with the property.

$$L = \int £ dx$$

and as a result, for the present problem we have

$$£ = ½ \left[\mu\dot{\varphi}^2 - T\left(\frac{\partial\varphi}{\partial x} \right)^2 \right] \qquad (34)$$

A derivation like that of article 9.2 yields (in one dimension - see equ. 18)

$$\frac{d}{dt}\left(\frac{\partial £}{\partial\left(\frac{\partial\varphi}{\partial t} \right)} \right) + \frac{d}{dx}\left(\frac{\partial £}{\partial\left(\frac{\partial\varphi}{\partial x} \right)} \right) - \frac{\partial £}{\partial\varphi} = 0 \qquad (35)$$

From equ. (34),

$$\frac{\partial £}{\partial\left(\frac{\partial\varphi}{\partial t} \right)} = \mu\dot{\varphi}, \frac{\partial £}{\partial\left(\frac{\partial\varphi}{\partial x} \right)} = -T\frac{\partial\varphi}{\partial x}, \frac{\partial £}{\partial\varphi} = 0$$

Substituting above into equ. (35), we get the familiar wave equation

$$\mu\ddot{\varphi} - T\frac{\partial^2\varphi}{\partial x^2} = 0$$

APPENDICES

A LABORATORY AND CENTRE OF MASS FRAMES

Consider a particle of mass m_1 moving with velocity $\vec{v}_1$ collides elastically with a particle of mass m_2, called target or scatterer, which is initially at rest. Such a frame is termed as the laboratory frame of reference (L-system-see Fig. A.1). The path of m_1 is deflected through an angle θ_1 after the collision. The velocities of the two particles m_1 and m_2 after collison are suppose, $\vec{v}_1{}'$ and $\vec{v}_2{}'$, respectively.

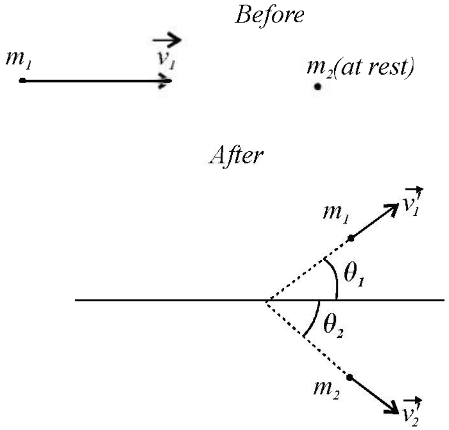

Fig. A.1 Laboratory frame

Let us denote the initial (before collision) velocities of the two particles by $\vec{u}_1$ and $\vec{u}_2$ in the reference frame in which the centre of mass (com) of the two particles is at rest. Such a frame is termed as the centre of mass frame of reference (C-system (see fig. A.2) - Frame moving with the velocity of the com w.r.t. L-system). Let the final velocities (after collision) of the two particles are $\vec{u}_1'$ and $\vec{u}_2'$, respectively. The velocities in the L-system are related to those in the C-system by

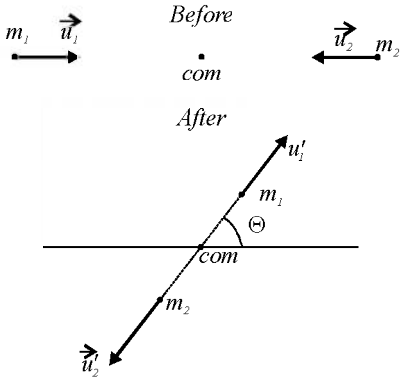

Fig. A.2 Centre of mass frame

$$\vec{\upsilon}_1 = \vec{u}_1 + \vec{V}, \quad \vec{\upsilon}_2 = \vec{u}_2 + \vec{V}$$

$$\vec{\upsilon}'_1 = \vec{u}'_1 + V, \quad \vec{\upsilon}'_2 = \vec{u}'_2 + V \tag{A.1}$$

where $\vec{V} = V\hat{x}$ is velocity of the com relative to the L-system, given by

$$\dot{\vec{R}}_{cm} = \vec{V} = \frac{m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}\vec{\upsilon}_1 \text{ as } \dot{\vec{r}}_2 = \vec{\upsilon}_2 = 0 \tag{A.2}$$

The conservation laws allow us to understand immediately certain characteristics of the collision. (i) Momentum conservation in C - system (total linear momentum is zero) requires that the scattering angle of particle 1 equals that of the particle 2, that is, the trijectories must be collinear before and after the collision. If this is not so the com would not remain at rest after collision. (ii) if the kinetic energy is to be conserved, the magnitudes of velocities or speeds of particles in com frame must not be changed, so that

$$u'_1 = u_1 , u'_2 = u_2 \tag{A.3}$$

in an elastic collision.

Turning again to L- and C-systems (see Figs. A.3 and A.4),

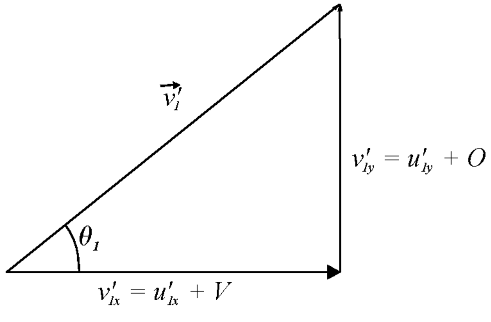

Fig. A.3 Laboratory frame showing components of $\vec{\upsilon}'_1$

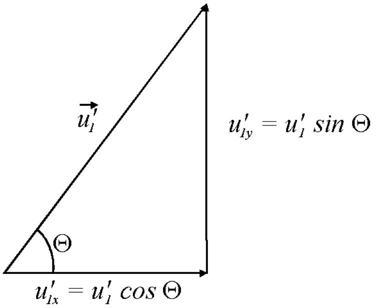

Fig. A.4 Centre of mass frame showing components of $\vec{u}_1'$

$$\tan\theta_1 = \sin\theta_1/\cos\theta_1 = \vec{\upsilon}_1' \sin\theta_1/\vec{\upsilon}_1' \cos\theta_1$$

$$= \frac{\upsilon_{1y}' = u_{1y}' + 0 = u_1' \sin\Theta}{\upsilon_{1x}' = u_{1x}' + V = u_1' \cos\Theta + V} \tag{A.4}$$

where we have used the fact that the y - component of the final velocity of particle 1 is identical in the two reference frames. Further since $\vec{u}_1' = u_1$ from (A. 3), we get

$$\tan\theta_1 = \frac{\sin\Theta}{\cos\Theta + V/u_1} \tag{A.5}$$

Now from equations (A.2) and (A.1),

$$\vec{V} = \frac{m_1}{m_1 + m_2}\left(\vec{u}_1 + \vec{V}\right)$$

$$\vec{V} = m_1/m_2\, \vec{u}_1 \tag{A.6}$$

Substituting (A.6) into (A. 5), one gets

$$\tan\theta_1 = \frac{\sin\Theta}{\cos\Theta + m_1/m_2} \tag{A.7}$$

B LEGENDRE TRANSFORMATION

Consider a function $f(x, y)$ of only two variables, so that the total differential of f has the form

$$df = \partial f/\partial x\, dx + \partial f/\partial y\, dy = u\, dx + v\, dy \tag{B.1}$$

where $$u = \partial f/\partial x, v = \partial f/\partial y \tag{B.2}$$

We wish now to change the basis from x, y to a new set of independent variables u, y. Let g be a function of u and y defined by the equation.

$$g = f - ux \tag{B.3}$$

The total differential of g is then given by

$$dg = df - udx - x\, du$$

which on using equ. (B.1) becomes

$$dg = v\,dy - x\,du$$

and is exactly in the form desired. Writing

$$dg = \partial g/\partial y\,dy + \partial g/\partial u\,du,$$

we find

$$x = -\partial g/\partial u\,,\; v = \partial g/\partial y \tag{B.4}$$

which are the analogues of equs. (B.2).

Thus if $u = \partial f/\partial x$, then

$$g(u, y) = f(x, y) - ux \tag{B.5}$$

C RELATIVISTIC LAGRANGIAN AND HAMILTONIAN

Let us try to find a Lagrangian function L for which the Lagrange equations as obtained from Hamilton's principle

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0 \tag{C.1}$$

agree with the known relativistic equation of motion,

$$\frac{dp_i}{dt} = F_i \tag{C.2}$$

where F_i is the ith component of the force and the momentum p_i indicated in the above equation is not m v_i, but rather some relativistic generalization that reduces to it in the limit of small velocities.

We suppose that a suitable relativistic Lagrangian for a single particle acted on by a conservative force field with potential energy V independent of velocity (depending only on the position) is

$$L = -m_0 c^2 \sqrt{1 - v^2/c^2} - V \tag{C.3}$$

where v is the speed of the particle in the Lorentz (Inertial) frame under consideration. That this is the correct Lagrangian can be shown by demonstrating that the resultant Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial x_i} = 0,$$

agree with equ. (C.2) because the potential energy V is velocity independent and $v_i = \dot{x}_i$ occurs only in the first term of (C.3) and therefore,

$$\frac{\partial L}{\partial v_i} = -m_0 c^2 \times \frac{-\frac{1}{2} 2v_i/c^2}{\sqrt{1 - v^2/c^2}} = \frac{m_0 v_i}{\sqrt{1 - v^2/c^2}} = p_i \tag{C.4}$$

which when substituted in (C.3) gives

$$\frac{d}{dt}\frac{m_0 v_i}{\sqrt{1 - v^2/c^2}} = \frac{d}{dt}p_i = -\frac{\partial V}{\partial x_i} = F_i \tag{C.5}$$

and this is nothing but the relativistic equation of motion. Therefore, the choice of L represented by equ. (C.3) is correct. One may note that the Lagrangian L is no longer T-V but that the partial derivative of L with velocity is still the momentum (relativistic).

We have shown earlier that if L does not contain time explicitly, there exists a constant of motion, the Hamiltonian.

$$H = \sum_i p_i \dot{q}_i - L$$

in which $\sum_i$ denotes the sum of over all components.

Truly speaking it is still true as we would show below. For the given case of a single particle†,

$$H = \sum_i p_i \dot{q}_i - L = \sum_i \frac{m_0 \upsilon_i^2}{\sqrt{1-\upsilon^2/c^2}} + m_0 c^2 \sqrt{1-\upsilon^2/c^2} + V \qquad \text{(C.6)}$$

where we have used equs. (C.4), (C.3) and $\dot{q}_i = \dot{x}_i = \upsilon_i$ amounts to

$$H = \frac{m_0(\upsilon_x^2 + \upsilon_y^2 + \upsilon_z^2)}{\sqrt{1-\upsilon^2/c^2}} + m_0 c^2 \sqrt{1-\upsilon^2/c^2} + V$$

$$= \frac{m_0 c^2}{\sqrt{1-\upsilon^2/c^2}} + V$$

$$= mc^2 + V$$

$$= m_0 c^2 + T + V = E = \text{Constant} \qquad \text{(C.7)}$$

where we have used the fact that $\upsilon^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \upsilon_x^2 + \upsilon_y^2 + \upsilon_z^2$. Therefore, the relativistic Hamiltonian is equal to the total energy, but it must include the rest mass energy as well. T of course, is the relativistic kinetic energy and mc^2 is defined as the relativistic energy of a free particle

D GEOMETRICAL OPTICS AND WAVE MECHANICS

Let us consider conservative systems for which the Hamiltonian is a constant of motion and is identical with the total energy. In such a case (see equs. 12, 25 - Chapter 8) Hamilton's principal function

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} (T-V)dt = \int_{t_1}^{t_2} 2T - (T+V) = \int_{t_1}^{t_2} (2T-E)dt$$

$$= W - E(t_2 - t_1) \qquad \text{(D.1)}$$

as $$H = \sum_i p_i \dot{q}_i - L$$

or $$\sum_i p_i \dot{q}_i = H + L = T + V + T - V = 2T$$

or $$\int \sum_i p_i \dot{q}_i dt = \int \sum_i p_i dq_i = W = \int 2T dt$$

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† In this case of single particle $i = 1, 2, 3$. $x_1 = x, x_2 = y, x_3 = z$ and $\upsilon_1 = \upsilon_x, \upsilon_2 = \upsilon_y, \upsilon_3 = \upsilon_z$.

Taking $t_2 = t$, $t_1 = 0$, from equ. (D. 1), we get

$$S(q, t) = W(q) - Et \tag{D.2}$$

Further from equ. (11) - Chapter 8,

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i = \sum_i p_i\dot{q}_i - H \tag{D.3}$$

In view of equ. (D.2),

$$\frac{\partial}{\partial} = -E$$

and H being identical with E, from equ. (D.3), we get

$$p_i = \frac{\partial S}{\partial q_i} \quad (i = 1 \text{ to } s) \tag{D.4}$$

which is nothing but the generalized momentum in terms of S.

Taking our conservative system consisting of only a single particle and taking the generalized coordinates to be the cartesian coordinates, that is $q_i = x_i$ ($i = 1, 2, 3$), the configuration space then reduces to ordinary three - dimensional space. This in turn simplifies the problem. As a result, we get

$$p_i = \frac{\partial S}{\partial q_i} = \frac{\partial W}{\partial q_i} = \frac{\partial W}{\partial x_i} = (\text{grad } W)_i \; (i = 1, 2, 3) \tag{D.5}$$

where use has been made of equ. D. 2

The above together with the formula for the Hamiltonian.

$$1/2m \,|\text{grad } W|^2 + V = E$$

yields

$$|\text{grad } W| = \sqrt{2m(E - V)} \tag{D.6}$$

Since W, the characteristic function is independent of time, the surfaces of constant W in configuration space have fixed locations ($W(x, y, z)$ = constant represents a family of fixed surfaces). The principal function $S(x, y, z, t)$ = constant, of course, stands for a family of moving surfaces (S being function of time). A surface characterized by a constant value of S must coincide at a given time with some particular surface of constant W. However, the value of W corresponding to a definite value of S changes with time in accordance with equ. (D.2). Therefore, at $t = 0$ the surfaces of $S = a$ and $S = b$ coincide with the surfaces for which $W = a$ and $W = b$, respectively (see Fig. D.1). At a time dt later the surface $S = a$ now coincides with the surface for which $W = a + E$ dt and similarly $S = b$ is located at the surface $W = b + Edt$. In effect, in a time dt the surface $S = a$ have moved from $W = a$ to $W = a + Edt$. The motion of the surface in time is similar to the propagation of a wavefront. The surfaces of constant S may thus be considered as wave fronts propagating in configuration space.

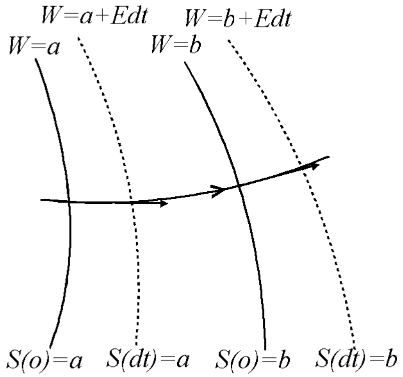

Fig. D.1 Geometry of fixed (W = const.) and moving (W + Edt = const.) surfaces propagating as a wavefront

The phase velocity of the wavefront (surface of constant S)

$$u = ds/dt$$

where ds is the perpendicular distance to which the wavefront has moved in infinitesimal time *dt*.

Now in time dt the S surface travels from a surface *W* to a new surface on which the value of the characteristic function is $W + dW$, where

$$dW = Edt \tag{D.7}$$

The change *dW* is also related to the normal distance ds as

$$dW = |\text{grad } W|ds = Edt \tag{D.8}$$

where use of equ. (D. 7) has been made.

Therefore,

$$u = ds/dt = \frac{E}{|\,grad\ W\,|}$$

which on using equ. (D. 6) results in

$$u = \frac{E}{\sqrt{2m(E-V)}} = \frac{E}{\sqrt{2mT}} = \frac{E}{p} = E/mv \tag{D.9}$$

where *T* is the kinetic energy, *p* the momentum and *v* the spatial velocity of the particle. Equ. (D. 9), therefore states that the velocity of a point on a surface of constant *S* is inversely proportional to the spatial velocity of the particle whose motion is being described by *S*.

To study the nature of waves whose fronts are surfaces of constant *S*, one must find certain characteristic quantities, such as frequency and wavelength. These can be obtained by making an analogy with the propagation of light waves, whose equation is

$$\nabla^2\Psi - n^2/c^2\ \frac{\partial^2 \psi}{\partial t^2} = 0 \tag{D.10}$$

where n is the refractive index of the medium. In general, n will be a function of position in space. If n = constant, equ. (D. 10) is satisfied by a plane wave solution of the form

$$\Psi = \Psi_0\,(x, y, z)\, e^{i(\bar{k}.\bar{r}-\omega t)} \tag{D.11}$$

where the wave number k and the frequency ω are connected by the relation

$$k = 2\pi/\lambda = m\omega/c = nk_0 \tag{D.12}$$

Taking the direction of $\vec{k}$ along the positive z-axis, we have

$$\Psi = \Psi_0\ e^{ik_0(nz-ct)} \tag{D.13}$$

where k_0 is the wave number in vacuum. We shall be interested, however, in the case of geometrical optics, where n is no longer constant but varies slowly in space. The plane wave is then no longer a solution of the wave equation (D. 10), the variation of n with position would distort and bend the wave. Since n is assumed to change only slowly in space, we seek a solution resembling the plane wave as closely as possible, that is, the solution of D. 10 is close to the form

$$\Psi = \Psi_0\,(x, y, z)\, e^{ik_0(L(x,y,z)-ct)} \tag{D.14}$$

where L is called eikonal. If n = constant, then $L = nz$ called the optical path length. Introducing (D. 14) into (D.10), evaluating the derivatives and then separating the real and imaginary parts, we get

$$\psi_0^2 k_0^2\,[n^2 - |\text{grad } L|^2] + \nabla^2\,\Psi_0 = 0 \tag{D.15}$$

$$\Psi_0\,\nabla^2 L + 2(\text{grad } \Psi_0)\cdot(\text{grad } L) = 0 \tag{D.16}$$

Since n varies only slowly with distance, that is, n does not change greatly over distances of the order of wavelength, effectively means that the wavelength is small compared to the dimension of any change in medium or the distance on which the medium displays its non-homogeneity (which is the assumption of geometrical optics). Then the presence of $k_0^2 = 4\pi^2/\lambda_0^2$ makes the first term in equ. (D.15) much greater than the second, resulting in

$$|\text{grad } L| = n \tag{D.17}$$

The above equation is known as the eikonal equation of geometrical optics. The surfaces of constant L determined by this equation are the surfaces of constant optical phase and therefore, define the wavefronts. The equal -phase surfaces are given by

$$f(x, y, z, t) = k_0\,L(x, y, z) - \omega t = \text{const.} \tag{D.18}$$

Comparing (D. 6) with the eikonal equation (D.17), we conclude that they are similar: the quantity $\sqrt{2m(E-V)}$ plays the role of n and the characteristic function W that of L, while the surfaces S = Const. are analogous to the surfaces f = const.

Therefore, we can set

$$S = \alpha f,\ W = \alpha k_0\, L(x, y, z),\ E = \alpha\omega \text{(see D.2)} \qquad \text{(D.19)}$$

where α is an arbitrary constant. Hence

$$n = |\text{grade } L| = 1/\alpha k_0\, |\text{grad } W| = c/\alpha\omega\ \sqrt{2m(E-V)} \qquad \text{(D.20)}$$

This analogy shows that the propagation of constant phase waves and light waves are similar phenomenon.

Further putting the value of n from equ. (D.20) in equ. (D. 10), we get

$$\nabla^2 \Psi - n^2/c^2\, \partial^2\psi/\partial t^2 = \nabla^2\Psi - n^2/c^2\, \omega^2 \psi = \nabla^2 \Psi - \frac{2m(E-V)}{\alpha^2}\Psi = 0$$

Taking the constant $\alpha = \hbar$, we get

$$\nabla^2\Psi - \frac{2m(E-V)}{\hbar^2}\Psi = 0 \qquad \text{(D.21)}$$

which is Schrödinger equation for the single particle in a conservative field. This shows that the classical mechanics is only a geometrical optics approximation (wavelength small compared to the distance on which the medium displays non-homogeneity).

Further if we apply the principle of least action in Jacobi's form - see Sec. 6.5 equ. (29), that is,

$$\Delta \int_{P_1}^{P_2} \sqrt{2m(E-V)}\, ds = 0 \qquad \text{(D.22)}$$

for a single particle, P_1 and P_2 being the end points at times t_1 and t_2 in the real space $\left(T = ½\sum_i m_i |d\vec{r}_i / dt|^2 = ½\sum_i m_i (ds_i / dt)^2\right)$. Introducing this into equ. (29) of sec. 6.5, we get

$$\Delta \int_{P_1}^{P_2} \sqrt{2m(E-V)\sum_i m_i\, ds_i^2} = 0$$

For single particle the above statement reduces to (D. 22) and observe that according to (D. 20) the integral is proportional to n, we get

$$\Delta \int n\, ds = 0 \qquad \text{(D.23)}$$

which is nothing but the well known Fermat's principle of geometrical optics (or of least optical path for trajectories of light rays)

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